

# A System $F$ accounting for scalars

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## Abstract

The algebraic  $\lambda$ -calculus [40] and the linear-algebraic  $\lambda$ -calculus [3] extend the  $\lambda$ -calculus with the possibility of making arbitrary linear combinations of  $\lambda$ -calculus terms (preserving  $\sum \alpha_i \cdot \mathbf{t}_i$ ). In this paper we provide a fine-grained, System  $F$  -like type system for the linear-algebraic  $\lambda$ -calculus (*Lineal*). We show that this *scalar* type system enjoys both the subject-reduction property and the strong-normalisation property, which constitute our main technical results. The latter yields a significant simplification of the linear-algebraic  $\lambda$ -calculus itself, by removing the need for some restrictions in its reduction rules – and thus leaving it more intuitive. But the more important, original feature of this scalar type system is that it keeps track of ‘the amount of a type’ that this present in each term. As an example, we show how to use this type system in order to guarantee the well-definiteness of probabilistic functions ( $\sum \alpha_i = 1$ ) – thereby specializing *Lineal* into a probabilistic, higher-order  $\lambda$ -calculus. Finally we begin to investigate the logic induced by the *scalar* type system, and prove a no-cloning theorem expressed solely in terms of the possible proof methods in this logic. We discuss the potential connections with Linear Logic and Quantum Computation.

*Keywords:* Probabilistic calculus, Quantum logic, Linear logic

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## 1. Introduction

**(Linear-)Algebraic  $\lambda$ -calculi.** A number of recent works seek to endow the  $\lambda$ -calculus with a vector space structure. This agenda has emerged simultaneously in two different contexts.

- The exponential-free fragment of *Linear Logic* is a logic of resources where the propositions themselves stand for those resources – and hence cannot be discarded nor copied. When seeking to find models of this logic, one obtains a particular family of vector spaces and differentiable functions over these. It is by trying to capture back these mathematical structures into a programming language that Ehrhard and Regnier have defined the *differential  $\lambda$ -calculus* [16], which has an intriguing differential operator as a built-in primitive, and some notion of module of the  $\lambda$ -calculus terms, over the natural numbers. Lately Vaux [40] has focused his attention on a ‘differential  $\lambda$ -calculus without differential operator’, extending the module to finitely splitting positive real numbers. He obtained a confluence result in this case, which stands even in the untyped setting. More recent works on this *Algebraic  $\lambda$ -calculus* tend to consider arbitrary scalars [15, 36].
- The field of *Quantum Computation*, considers that computers being physical systems, they may behave according to quantum theory. It proves that, if this is the case, novel, more efficient algorithms are possible [33, 22] – which have no classical counterpart. Whilst partly unexplained, it is nevertheless clear that the algorithmic speed-up arises by tapping into the parallelism granted to us ‘for free’ by the *superposition principle*; which states that if  $\mathbf{t}$  and  $\mathbf{u}$  are possible states of a system, then so is the formal linear combination of them  $\alpha \cdot \mathbf{t} + \beta \cdot \mathbf{u}$  (with  $\alpha$  and  $\beta$  some arbitrary complex numbers, and up

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to some renormalizing factor). The idea of a module of  $\lambda$ -terms over an arbitrary scalar field arises quite naturally in this context. This was the motivation behind the *Linear-algebraic  $\lambda$ -calculus* by Dowek and one of the authors, who obtained a confluence result which holds for arbitrary scalars, and again covers the untyped setting.

These two languages are rather similar: they both merge higher-order computation, be it terminating or not, in its simplest and most general form (namely the untyped  $\lambda$ -calculus) together with linear algebra in its simplest and most general form also (the axioms of vector spaces). In fact they can simulate each other [4, 11]. Our starting point will be the second one, also referred to as *Lineal* in this work: because its confluence proof allows arbitrary scalars and because one has to make a choice.

**Other motivations to study (Linear-)Algebraic  $\lambda$ -calculi.** The two languages are also reminiscent of other works in the literature:

- *Algebraic and symbolic computation.* The functional style of programming is based on the  $\lambda$ -calculus together with a number of extensions, so as to make everyday programming more accessible. Hence since the birth of functional programming there has been several theoretical studies of extensions of the  $\lambda$ -calculus in order to account for basic algebra (see for instance Dougherty’s algebraic extension [13] for normalising terms of the  $\lambda$ -calculus) and other basic programming constructs such as pattern-matching, together with the sometimes non-trivial associated type theories (see for instance Petit’s  $\lambda$ -calculus extension and type system [28] with pattern matching). Whilst this was not the original motivation behind (linear-)algebraic  $\lambda$ -calculi, these languages could still be viewed as just an extension of the  $\lambda$ -calculus in order to handle operations over vector spaces, and make everyday programming more accessible upon them. The main difference in approach is that here the  $\lambda$ -calculus is not seen as a control structure which sits on top of the vector space data structure, controlling which operations to apply and when. Rather, the  $\lambda$ -calculus terms themselves can be summed and weighted, hence they actually are the basis of the vector space... upon which they can also act.
- *Parallel and probabilistic computation.* This intertwining of concepts is essential if seeking to represent parallel or probabilistic computation as it is the computation itself which must be endowed with a vector space structure. The ability to superpose  $\lambda$ -calculus terms in that sense takes us back to Boudol’s parallel  $\lambda$ -calculus [8], and may also be viewed as taking part of a wave of probabilistic extensions of calculi, *e.g.* [9, 23, 10].

Hence (linear-)algebraic  $\lambda$ -calculi can be seen as a platform for various applications, ranging from algebraic computation, probabilistic computation, quantum computation and resource-aware computation.

**The existence of a norm in (Linear-)Algebraic  $\lambda$ -calculi.** We will not be developing any of the above-mentioned applications specifically in this paper. In the same way that the theory of vector spaces has many applications, but has also got many theoretical refinements that deserve to be studied in their own right, we take the view the theory of vector spaces plus  $\lambda$ -calculus has got theoretical refinements that deserve to be studied in their own right. Moreover, these theoretical refinements are often necessary in order to address the applications, as is notoriously the case for instance with the notion of norm. This is the case again here, for instance if we want to be able to interpret a linear combination of terms  $\sum \alpha_i \cdot \mathbf{t}_i$  as a probability distribution, we will need to make sure that it has norm one. The same is true if we want to interpret  $\sum \alpha_i \cdot \mathbf{t}_i$  as quantum superposition, but with a different norm <sup>1</sup>. Yet the very definition of a norm is difficult in our context: deciding whether a term terminates is undecidable; but these terms produce infinities, hence convergence of the norm is undecidable. Related to this precise topic, Vaux has studied simply typed algebraic  $\lambda$ -calculus, ensuring convergence of the norm [40]. Following his work, Tasson has studied some model-theoretic properties of the *barycentric* ( $\sum \alpha_i = 1$ ) subset of this simply typed calculus

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<sup>1</sup>Whereas it is clear already that *Lineal* is a quantum  $\lambda$ -calculus, in the sense that any quantum algorithm can be expressed in this language [3], the converse, alas, is not true, in the sense that some programs in *Lineal* express evolutions which are not valid quantum algorithms. This is precisely because *Lineal* does not restrict its vectors to be normalized and its applications to be unitary.

[36]. In the process of revising this paper we have also become aware of some just published work by Ehrhard, which proves the convergence of a Taylor series expansion of algebraic  $\lambda$ -calculus terms, via a System  $F$  typing system [15].

Hence, standard type systems provide part of the solution: they ensure the convergence of (the existence of) the norm of a term. And indeed it is no so hard to define a simple extension of System  $F$  that fits *Lineal* – just by providing the needed rules to type additions, scalar products and the null vector in some trivial manner (see definition 5). In this paper we provide a full-blown proof of strong normalisation from this type system (see section 5). A byproduct of this result is that we are able to remove several conditions that were limiting the reduction rules of *Lineal*, because their purpose was really to keep indefinite form from reducing (such as  $\mathbf{t} - \mathbf{t}$ , with  $\mathbf{t}$  not normal and hence potentially infinite). This is a significant contribution, as it makes *Lineal* into a simpler language.

**Quantified types for (Linear-)Algebraic  $\lambda$ -calculi.** Standard type systems only provide a part of the solution; they are unable for instance to impose upon the language that any well-typed linear combination of terms  $\sum \alpha_i \mathbf{t}_i$  has  $\sum \alpha_i = 1$ . That is unless we provide them with a handle upon these scalars. This is the purpose of the *scalar* type system which is proposed in this paper. Our contribution in this paper is indeed a type system which manages to keep track of ‘the amount of a type’ by summing the amplitudes of its contributing terms, and reflects this amount within the type. As an example of its use, we have demonstrated that this provides a type system which guarantees well-definiteness of probabilistic functions (see section 6) in the sense that it specializes *Lineal* into a probabilistic, higher-order  $\lambda$ -calculus. This example also illustrates how further advances in the theory may help specialize (linear-)algebraic  $\lambda$ -calculi into more specific-purpose languages, closer to applications – each of them requiring its own type system. Endowing *Lineal* with a non-trivial, more informative / fine-grained type system was quite a challenge, as the reader will judge. But we believe that fine-grained type theories for these non-deterministic / parallel / probabilistic extensions of calculi, which would capture how many processes are in what class of states, with what probability... may eventually lead to interesting forms of quantitative logics. In this paper we only begin to explore that route, by proving a *no-cloning theorem* for the scalar logic induced by the scalar type system – which echoes a long literature on Linear Logic and Quantum Computation.

Section 2 presents an overview of the linear-algebraic  $\lambda$ -calculus (*Lineal*) [3]. Section 3 presents the *scalar* type system with its grammar, equivalences and inference rules. Section 4 shows the subject reduction property giving consistency to the system. Section 5 shows the strong normalisation property for this system, allowing us to lift the above discussed restrictions in the reduction rules. In section 6 we formalise the type system  $\mathcal{P}$  for probabilistic calculi and work out the no-cloning theorem in the logic induced by the *scalar* type system. In Section 7 we present a discussion and future work. Section 8 concludes.

## 2. Linear-algebraic $\lambda$ -calculus

*Intuitions.* As a language of terms, *Lineal* is just  $\lambda$ -calculus together with the possibility to make arbitrary linear combinations of terms  $(\alpha.\mathbf{t} + \beta.\mathbf{u})$ . In terms of operational semantics, *Lineal* merges higher-order computation, be it terminating or not, in its simplest and most general form (the  $\beta$ -reduction of the untyped  $\lambda$ -calculus) together with linear algebra in its simplest and most general form also (the oriented axioms of vector spaces). Care must be taken, however, when merging these two families of reduction rules. For instance the term  $(\lambda x \mathbf{x} \otimes \mathbf{x}) (\alpha.\mathbf{t} + \beta.\mathbf{u})$ , where  $\otimes$  stands for the usual encoding of the tuple, maybe thought of as reducing to  $(\alpha.\mathbf{t} + \beta.\mathbf{u}) \otimes (\alpha.\mathbf{t} + \beta.\mathbf{u})$  in a call-by-name-oriented view, or to  $\alpha.(\mathbf{t} \otimes \mathbf{t}) + \beta.(\mathbf{u} \otimes \mathbf{u})$  in a call-by-value-oriented view, also compatible with the view that application should be bilinear (cf. *Application rules*, below). Leaving both options open would break confluence, the second option was chosen, which entails restricting the  $\beta$ -reduction to terms not containing sums or scalars in head position (cf. *Beta reduction* rule, below).

Instead introducing vector spaces via an oriented version of their axioms (e.g.  $\alpha.\mathbf{u} + \beta.\mathbf{u} \rightarrow (\alpha + \beta).\mathbf{u}$ ), one could have decided to perform the  $\beta$ -reduction ‘modulo equality in the theory of vector spaces’ (e.g.  $\alpha.\mathbf{u} + \beta.\mathbf{u} \rightarrow (\alpha + \beta).\mathbf{u}$ ). But there is a good reason not to do that: It is possible to define fixed point

operators

$$\mathbf{Y} = \lambda y ((\lambda x (y + (x x))) (\lambda x (y + (x x))))$$

and a term  $\mathbf{b}$  such that  $(\mathbf{Y} \mathbf{b})$  reduces to  $\mathbf{b} + (\mathbf{Y} \mathbf{b})$  and so on. Modulo equality over vector spaces, the theory would be inconsistent, as the term  $(\mathbf{Y} \mathbf{b}) - (\mathbf{Y} \mathbf{b})$  would then be equal to  $\mathbf{0}$ , but would also reduce to  $\mathbf{b} + (\mathbf{Y} \mathbf{b}) - (\mathbf{Y} \mathbf{b})$  and hence also be equal to  $\mathbf{b}$ . Instead, this problem can be fixed by restricting rules such as  $\alpha.\mathbf{u} + \beta.\mathbf{u} \rightarrow (\alpha + \beta).\mathbf{u}$  to terms that cannot reduce forever (cf. *Factorization rules*, below), matching the old intuition that indefinite forms ‘ $\infty - \infty$ ’ must be left alone. Moreover, oriented axioms of vector spaces define vector spaces, and no more than vector spaces, just as well as the original axioms do, as was shown in [3]. Plus the orientation serves a purpose: it presents the vector in its canonical form.

*Definitions.* Consider a first-order language, called *the language of scalars*, containing at least constants 0 and 1 and binary function symbols  $+$  and  $\times$ . Then the *language of vectors* is a two-sorted language, with a sort for vectors and a sort for scalars. The sort for vectors is described by the following term grammar:

$$\begin{aligned} \mathbf{t} &::= \mathbf{b} \mid (\mathbf{t} \mathbf{t}) \mid \mathbf{0} \mid \alpha.\mathbf{t} \mid \mathbf{t} + \mathbf{t} \\ \mathbf{b} &::= x \mid \lambda x \mathbf{t} \end{aligned}$$

where  $\alpha$  has the sort of scalars. Those scalars may themselves be defined by a term grammar, and endowed with a term rewrite system (TRS) which is compatible with their basic ring operations  $(+, \times)$ . Formally it is captured in the definition [3, sec. III – def. 1] of a scalar rewrite system, but for our purpose it is sufficient to think of them as a ring. We reproduce this definition for completeness.

**Definition 1 (Scalar rewrite system).** A scalar rewrite system  $S$  is an arbitrary rewrite system defined on scalar terms and such that

- $S$  is terminating and confluent on closed terms,
- for all closed terms  $\alpha, \beta$  and  $\gamma$ , the pair of terms
  - $0 + \alpha$  and  $\alpha$ ,  $0 \times \alpha$  and  $0$ ,  $1 \times \alpha$  and  $\alpha$ ,
  - $\alpha \times (\beta + \gamma)$  and  $(\alpha \times \beta) + (\alpha \times \gamma)$ ,
  - $(\alpha + \beta) + \gamma$  and  $\alpha + (\beta + \gamma)$ ,  $\alpha + \beta$  and  $\beta + \alpha$ ,
  - $(\alpha \times \beta) \times \gamma$  and  $\alpha \times (\beta \times \gamma)$ ,  $\alpha \times \beta$  and  $\beta \times \alpha$
 have the same normal forms,
- 0 and 1 are normal terms.

More importantly there are 16 rewrite rules for vectors, modulo associativity and commutativity, that is an *AC-rewrite system* [24], divided in four groups:

<i>Elementary rules:</i>	<i>Factorisation rules:</i>	<i>Application rules:</i>	<i>Beta reduction:</i>
$\mathbf{u} + \mathbf{0} \rightarrow \mathbf{u},$	$\alpha.\mathbf{u} + \beta.\mathbf{u} \rightarrow (\alpha + \beta).\mathbf{u}$	$(\mathbf{u} + \mathbf{v}) \mathbf{w} \rightarrow (\mathbf{u} \mathbf{v}) + (\mathbf{u} \mathbf{w})$	$(\lambda x \mathbf{t}) \mathbf{b} \rightarrow \mathbf{t}[\mathbf{b}/x]$
$0.\mathbf{u} \rightarrow \mathbf{0},$	$(*)$ ,	$(**)$ ,	$(***)$ .
$1.\mathbf{u} \rightarrow \mathbf{u},$	$\alpha.\mathbf{u} + \mathbf{u} \rightarrow (\alpha + 1).\mathbf{u}$	$\mathbf{w} (\mathbf{u} + \mathbf{v}) \rightarrow (\mathbf{w} \mathbf{u}) + (\mathbf{w} \mathbf{v})$	
$\alpha.\mathbf{0} \rightarrow \mathbf{0},$	$(*)$ ,	$(**)$ ,	
$\alpha.(\beta.\mathbf{u}) \rightarrow (\alpha \times \beta).\mathbf{u},$	$\mathbf{u} + \mathbf{u} \rightarrow (1 + 1).\mathbf{u}$	$(\alpha.\mathbf{u}) \mathbf{v} \rightarrow \alpha.(\mathbf{u} \mathbf{v})$	$(*)$ ,
$\alpha.(\mathbf{u} + \mathbf{v}) \rightarrow \alpha.\mathbf{u} + \alpha.\mathbf{v}.$	$(*)$ .	$\mathbf{v} (\alpha.\mathbf{u}) \rightarrow \alpha.(\mathbf{v} \mathbf{u})$	$(*)$ ,
		$\mathbf{0} \mathbf{u} \rightarrow \mathbf{0},$	
		$\mathbf{u} \mathbf{0} \rightarrow \mathbf{0}.$	

where  $+$  is an associative-commutative (AC) symbol and

- (\*) these rules apply only if  $\mathbf{u}$  is a closed normal term.
- (\*\*) these rules apply only if  $\mathbf{u} + \mathbf{v}$  is a closed normal term.
- (\*\*\*) the rule apply only when  $\mathbf{b}$  is a base term.

Restriction (\*\*\*) is the one that limits the beta reduction, whereas restrictions (\*) and (\*\*) are those that avoid confluence problems related to infinities and indefinite forms, as discussed above.

### 3. The *Scalar* Type System

We now introduce our *scalar* type system for *Lineal*.

The language of types is defined by the following abstract grammar:

$$\mathcal{T} = \mathcal{U} \mid \forall X. \mathcal{T} \mid \alpha. \mathcal{T} \mid \bar{0}$$

$$\mathcal{U} = X \mid \mathcal{U} \rightarrow \mathcal{T} \mid \forall X. \mathcal{U}$$

where  $\alpha \in \mathcal{S}$  and  $(\mathcal{S}, +, \times)$  is any commutative ring. Notice that the grammar for  $\mathcal{U}$ , which we call *unit types*, does not allow for scalars except to the right of an arrow. Notice also the novelty of having scalars *weighting* the amount of a type.

We also define an equivalence between types as follows:

**Definition 2.** Let  $\alpha, \beta \in \mathcal{S}$  and  $T \in \mathcal{T}$ . We define the type equivalence  $\equiv$  to be the least congruence such that

$$\bullet \alpha.\bar{0} \equiv \bar{0} \quad \bullet 0.T \equiv \bar{0} \quad \bullet 1.T \equiv T \quad \bullet \alpha.(\beta.T) \equiv (\alpha \times \beta).T \quad \bullet \forall X. \alpha.T \equiv \alpha. \forall X.T$$

and extend this definition to equivalence between sequents in the following way:  $[\Gamma \vdash t:T \equiv \Gamma \vdash t:S] \Leftrightarrow [T \equiv S]$ .

Splitting the grammar into general types and unit types is a necessary consequence of the fact that we want scalars in the types to reflect scalars in the terms (e.g.  $\alpha.\lambda x \mathbf{t}$  should have the type  $\alpha.U$ ). Indeed if we did not have the restriction on the left side of an arrow being a unit type, i.e.  $\mathcal{U} \rightarrow \mathcal{T}$ , then we would have things like  $(\alpha.X) \rightarrow X$ , which *a priori* do not make sense, because abstractions receive only base terms as arguments. This can be fixed by adding the equivalence  $(\alpha.A) \rightarrow B \equiv \alpha.(A \rightarrow B)$ , making sure that  $\alpha$  is non-zero. But still we would need to keep the  $\rightarrow E$  rule restricted to having a unit type on the left of the arrow, otherwise we break the required correspondence between scalars-in-types and scalars-in-terms, e.g.:

$$\frac{\vdash \alpha.\lambda x x : (\alpha.T) \rightarrow T \quad \vdash \mathbf{t} : \alpha.T}{\vdash (\alpha.\lambda x x) \mathbf{t} : T} \quad \text{noticing that } (\alpha.\lambda x x) \mathbf{t} \rightarrow^* \alpha.\mathbf{t}$$

Regarding typing rules, as we just said, we want the scalars in the types to represent those in the terms. Hence we want a rule as follows:

$$\frac{\Gamma \vdash \mathbf{u} : T}{\Gamma \vdash \alpha.\mathbf{u} : \alpha.T} \alpha I$$

We need also need to take care of sums of terms:

$$\frac{\Gamma \vdash \mathbf{u} : \alpha.T \quad \Gamma \vdash \mathbf{v} : \beta.T}{\Gamma \vdash \mathbf{u} + \mathbf{v} : (\alpha + \beta).T} +I$$

The term  $\mathbf{0}$  should have type  $0.T$  for any type, as it is the result of, for example, terms like  $\mathbf{t} - \mathbf{t}$ . Equivalences between types also give us  $0.T \equiv \bar{0}$ , so we add this as an axiom:

$$\frac{}{\Gamma \vdash \mathbf{0} : \bar{0}} ax\bar{0}$$

Finally, let us go back to the application. The standard rule  $\rightarrow E$  needs to be made consistent with the extra rules for application that we have apart from beta reduction; namely the *Application rules*:

1.  $(\mathbf{u} + \mathbf{v}) \mathbf{w} \rightarrow (\mathbf{u} \mathbf{v}) + (\mathbf{v} \mathbf{w})$
2.  $\mathbf{w} (\mathbf{u} + \mathbf{v}) \rightarrow (\mathbf{w} \mathbf{u}) + (\mathbf{w} \mathbf{v})$
3.  $(\alpha.\mathbf{u}) \mathbf{v} \rightarrow \alpha.(\mathbf{u} \mathbf{v})$
4.  $\mathbf{v} (\alpha.\mathbf{u}) \rightarrow \alpha.(\mathbf{v} \mathbf{u})$
5.  $\mathbf{0} \mathbf{u} \rightarrow \mathbf{0}$

## 6. $\mathbf{u} \mathbf{0} \rightarrow \mathbf{0}$

Notice that the terms  $\mathbf{u}$  and  $\mathbf{v}$  in rules 1 and 2 must now have the same type (up to a scalar), so the type of  $\mathbf{u} + \mathbf{v}$  is analogous to the type of  $\alpha.\mathbf{u}$  in rules 3 and 4. Also, the type for  $\mathbf{0}$  in rules 5 and 6 is the same as the type of  $\mathbf{0}.\mathbf{u}$ . So we can focus our discussion on rules 3 and 4.

By rule 3, we must have:

$$\frac{\Gamma \vdash \mathbf{u} : \alpha.(U \rightarrow T) \quad \Gamma \vdash \mathbf{v} : U}{\Gamma \vdash \mathbf{u} \mathbf{v} : \alpha.T}$$

By rule 4, we must have:

$$\frac{\Gamma \vdash \mathbf{u} : U \rightarrow T \quad \Gamma \vdash \mathbf{v} : \beta.U}{\Gamma \vdash \mathbf{u} \mathbf{v} : \beta.T}$$

By combining these two we obtain:

$$\frac{\Gamma \vdash \mathbf{u} : \alpha.(U \rightarrow T) \quad \Gamma \vdash \mathbf{v} : \beta.U}{\Gamma \vdash \mathbf{u} \mathbf{v} : \alpha \times \beta.T} \rightarrow E$$

The complete set of typing rules is therefore System  $F$  with the changes and additions discussed above:

$$\begin{array}{c} \frac{}{\Gamma, x:U \vdash x:U} ax \quad \frac{\Gamma \vdash \mathbf{t}:T \quad T \equiv S}{\Gamma \vdash \mathbf{t}:S} \equiv \\[10pt] \frac{\Gamma \vdash \mathbf{u} : \alpha.(U \rightarrow T) \quad \Gamma \vdash \mathbf{v} : \beta.U}{\Gamma \vdash (\mathbf{u} \mathbf{v}) : (\alpha \times \beta).T} \rightarrow E \quad \frac{\Gamma, x:U \vdash \mathbf{t}:T}{\Gamma \vdash \lambda x \mathbf{t} : U \rightarrow T} \rightarrow I[U] \\[10pt] \frac{\Gamma \vdash \mathbf{u} : \forall X.T}{\Gamma \vdash \mathbf{u} : T[U/X]} \forall E[X := U] \quad \frac{\Gamma \vdash \mathbf{u} : T}{\Gamma \vdash \mathbf{u} : \forall X.T} \forall I[X] \text{ with } X \notin FV(\Gamma) \\[10pt] \frac{}{\Gamma \vdash \mathbf{0} : \bar{0}} ax_{\bar{0}} \quad \frac{\Gamma \vdash \mathbf{u} : \alpha.T \quad \Gamma \vdash \mathbf{v} : \beta.T}{\Gamma \vdash \mathbf{u} + \mathbf{v} : (\alpha + \beta).T} +I \quad \frac{\Gamma \vdash \mathbf{u} : T}{\Gamma \vdash \alpha.\mathbf{u} : \alpha.T} sI[\alpha] \end{array}$$

Where  $U \in \mathcal{U}$  and  $Name[Cond]$  represents a family of rules; one for each condition. Moreover,  $FV$  designates the set of free variables of a type, defined in the usual manner.

This fully specifies our *scalar* type system for *Lineal*. Notice that the scalars within the types reflect those of the contributing terms. The major part of our work will consist in proving properties about the system, such as subject reduction and strong normalisation.

## 4. Subject reduction

The following theorem ensures that typing is preserved by reduction, making our type system consistent. Having such a property is part of the basic requirements for a type system.

**Theorem 1 (Subject Reduction).** *Let  $t \rightarrow^* t'$ . Then  $\Gamma \vdash t : T \Rightarrow \Gamma \vdash t' : T$*

The proof of this theorem is quite long and non-trivial. This is the main technical contribution of the paper. In case the reader is not interested by the technical details, he may skip the remaining of this section and continue directly in section 5.

### 4.1. Preliminary lemmas

In order to prove this theorem, we need several auxiliary lemmas standing for general properties of our system. We have tried to provide an intuition of every lemma so as to make it easier to follow. Also, we divided them in four groups, reflecting the nature of their statement.

#### 4.1.1. Lemmas about types

The lemmas in this sub-subsection are statements about the properties of the types themselves, *i.e.* their equivalences.

It is not so hard to see that every type is equivalent to a scalar multiplied by a unit type (*i.e.* a type in  $\mathcal{U}$ ). Even a type in  $\mathcal{U}$  can always be multiplied by 1.

**Lemma 1 ( $\alpha$  unit).**  $\forall T \in \mathcal{T}, \exists U \in \mathcal{U}, \alpha \in \mathcal{S} \text{ s.t. } T \equiv \alpha.U$ .

PROOF. See appendix Appendix A.

This first lemma should not be misinterpreted however: this does not mean to say that any scalar appearing within a type can be factored out of the type. For example even a simple unit type  $X \rightarrow \alpha.X$  is not equivalent to  $\alpha.(X \rightarrow X)$ .

The following just says that when two types are equivalent, then the outer left scalars are the same:

**Lemma 2 (Unit does not add scalars).**  $\forall U, U' \in \mathcal{U}, \forall \alpha, \beta \in \mathcal{S}, \text{ if } \alpha.U \equiv \beta.U' \text{ then } \alpha = \beta \text{ and, if } \alpha \neq 0, \text{ then } U \equiv U'.$

PROOF. See appendix Appendix B.

Several of the following lemmas will be proved by induction on the size of the derivation tree, so, we need to formally define what we mean by size. In our definition we count the depth of the tree, but ignoring any application of an equivalence rule:

**Definition 3.** We define the *size of a derivation tree* inductively as follows

$$\text{size} \left( \frac{\frac{S'}{S} \equiv}{R} \right) = 0 \quad \text{size} \left( \frac{\frac{\pi_1}{S'} \frac{\pi_2}{S} R'}{\frac{S}{S} \equiv} \right) = \max\{\text{size}(\pi_1), \text{size}(\pi_2)\} + 1$$

where  $\pi_1, \pi_2$  are derivation trees,  $S$  is a sequent,  $R$  and  $R'$  are type inference rules, and  $S \equiv S'$ . Often we denote by  $S_n$  a sequent that can be derived with a proof of size  $n$ .

We will also need a concept of order between types. Without actually making a subtyping theory, we can define a partial order relation between types following [6]:

**Definition 4.**

1. Write  $A > B$  if either  $B \equiv \forall X.A$  or  $A \equiv \forall X.C$  and  $B \equiv C[U/X]$  for some  $U \in \mathcal{U}$ .
2.  $\geq$  is the reflexive and transitive closure of  $>$ .

**Remark 1.** This definition of an order is quite intuitive. The idea is that types in the numerator of

$$\frac{\Gamma \vdash \mathbf{t} : A}{\Gamma \vdash \mathbf{t} : \forall X.A} \forall I \text{ with } X \notin FV(\Gamma) \quad \text{or} \quad \frac{\Gamma \vdash \mathbf{t} : \forall X.C}{\Gamma \vdash \mathbf{t} : C[U/X]} \forall E$$

are greater than the types in the denominator, hence if  $\mathbf{t}$  is of a greater type, it must also be of the lesser type.

Notice that scalars do not interfere with the order, as stated by the following lemma:

**Lemma 3 (Scalars keep order).**  $T \geq T' \Rightarrow \alpha.T \geq \alpha.T'.$

PROOF. See appendix Appendix C.

The following lemma states that if two arrow types are ordered, then they are equivalent up to some substitution.

**Lemma 4 (Arrows comparison).**  $V \rightarrow R \geq U \rightarrow T \Rightarrow \exists \vec{W}, \vec{X} / U \rightarrow T \equiv (V \rightarrow R)[\vec{W}/\vec{X}]$

PROOF. See appendix Appendix D.

#### 4.1.2. Classic lemmas

The lemmas in this subsection are the classic ones, which appear in most subject reduction proofs.

As a pruned version of a subtyping system, we can prove the subtyping rule:

**Lemma 5 (Order typing).** *Let  $A \geq B$  and suppose no free type variable in  $A$  occurs in  $\Gamma$ . Then*

$$\Gamma \vdash u : A \Rightarrow \Gamma \vdash u : B$$

PROOF. *cf.* [6].

Proving subject reduction means proving that each reduction rule preserves the type. The way to do this is to go in the direction opposite to the reduction rule, *i.e.* to study the reduct so as to understand where it may come from, decomposing the redex in its basic constituents. Generation lemmas accomplish that purpose.

We will need four generation lemmas: the two classical ones, for applications (lemma 6) and for abstractions (lemma 7) and two new ones for the algebraic rules, one for products by scalars (lemma 8) and one for sums (lemma 9).

**Lemma 6 (Generation lemma (app)).** *Let  $S_n = \Gamma \vdash (\mathbf{u} \ \mathbf{v}) : \gamma.B$ . Then  $\exists \alpha, \beta \in \mathcal{S}, r, s \in \mathbb{N}_0, U \in \mathcal{U}$  and  $B' \in \mathcal{T}$  s.t.*

$$\left\{ \begin{array}{l} S_r = \Gamma \vdash \mathbf{v} : \alpha.U \\ S_s = \Gamma \vdash \mathbf{u} : \beta.U \rightarrow B' \\ B' \geq B \\ \gamma = \alpha \times \beta \\ \max(r, s) < n \end{array} \right.$$

PROOF. Induction over  $n$

*Basic case.*  $n = 1$ . We enumerate the four possible ways of deriving  $\Gamma \vdash \mathbf{u} \mathbf{v} : T$  in a derivation tree of size

1.  $S_r$  and  $S_s$  turn out as sub-trees.

$$\begin{array}{l}
1. \frac{\frac{\frac{\Gamma, x:U \rightarrow T, y:U \vdash x:1.U \rightarrow T}{\Gamma, x:U \rightarrow T, y:U \vdash x \ y:1.T} ax \text{ and } \equiv \quad \frac{\frac{\Gamma, x:U \rightarrow T, y:U \vdash y:1.U}{\Gamma, x:U \rightarrow T, y:U \vdash y \ y:1.T} ax \text{ and } \equiv}{\Gamma, x:U \rightarrow T, y:U \vdash x \ y:1.T} \rightarrow E \\
2. \frac{\frac{\frac{\Gamma, x:U \vdash x:1.U \rightarrow T}{\Gamma, x:U \rightarrow T \vdash x \ \mathbf{0}:0.T} ax \text{ and } \equiv \quad \frac{\frac{\Gamma, x:U \vdash \mathbf{0}:0.U}{\Gamma, x:U \rightarrow T \vdash \mathbf{0}:0.T} ax_{\overline{0}} \text{ and } \equiv}{\Gamma, x:U \rightarrow T \vdash x \ \mathbf{0}:0.T} \rightarrow E \\
3. \frac{\frac{\frac{\Gamma, y:U \vdash \mathbf{0}:0.U \rightarrow T}{\Gamma, y:U \vdash \mathbf{0} \ y:0.T} ax_{\overline{0}} \text{ and } \equiv \quad \frac{\frac{\Gamma, y:U \vdash y:1.U}{\Gamma, y:U \vdash y \ y:1.T} ax \text{ and } \equiv}{\Gamma, y:U \vdash \mathbf{0} \ y:0.T} \rightarrow E \\
4. \frac{\frac{\frac{\Gamma \vdash \mathbf{0}:0.U \rightarrow T}{\Gamma \vdash \mathbf{0} \ \mathbf{0}:0.T} ax_{\overline{0}} \text{ and } \equiv \quad \frac{\frac{\Gamma \vdash \mathbf{0}:0.U}{\Gamma \vdash \mathbf{0} \ \mathbf{0}:0.T} ax_{\overline{0}} \text{ and } \equiv}{\Gamma \vdash \mathbf{0} \ \mathbf{0}:0.T} \rightarrow E
\end{array}$$

*Inductive cases.* The possible cases are

1.  $\frac{\Gamma \vdash \mathbf{u}:\beta.(U \rightarrow B) \quad \Gamma \vdash \mathbf{v}:\alpha.U}{\Gamma \vdash \mathbf{u} \mathbf{v}:(\alpha \times \beta).B} \rightarrow E$  This is the trivial case.



2. 
$$\frac{\Gamma \vdash \mathbf{u} \mathbf{v} : \forall X. \gamma. B}{\Gamma \vdash \mathbf{u} \mathbf{v} : \gamma. B[U/X]} \forall E$$
 As  $\forall X. \gamma. B \equiv \gamma. \forall X. B$ , by the induction hypothesis  $\exists \alpha, \beta, r, s, U$  and  $B' \geq \forall X. B$  s.t.  $S_r = \Gamma \vdash \mathbf{u} : \beta. U \rightarrow B'$ ,  $S_s = \Gamma \vdash \mathbf{v} : \alpha. U$ ,  $\alpha \times \beta = \gamma$  and  $\max(r, s) < n - 1$ . As  $B' \geq \forall X. B > B[U/X]$  then by transitivity  $B' \geq B[U/X]$ .
3. 
$$\frac{\frac{\Gamma \vdash \mathbf{u} \mathbf{v} : \gamma. B}{\Gamma \vdash \mathbf{u} \mathbf{v} : \forall X. \gamma. B} \forall I}{\Gamma \vdash \mathbf{u} \mathbf{v} : \gamma. \forall X. B} \equiv$$
 by the induction hypothesis  $\exists \alpha, \beta, r, s, U$  and  $B' \geq B$  s.t.  $S_r = \Gamma \vdash \mathbf{u} : \beta. U \rightarrow B'$ ,  $S_s = \Gamma \vdash \mathbf{v} : \alpha. U$ ,  $\alpha \times \beta = \gamma$  and  $\max(r, s) < n - 1$ . By definition  $B > \forall X. B$ , so by transitivity  $B' \geq \forall X. B$ .

**Lemma 7 (Generation lemma (abs)).**  $\Gamma \vdash \lambda x \mathbf{t} : T \Rightarrow \exists U \in \mathcal{U}$  and  $A \in \mathcal{T}$  s.t.  $\Gamma, x : U \vdash \mathbf{t} : A$  and  $U \rightarrow A \geq T$ .

PROOF. Let  $S_n = \Gamma \vdash \lambda x \mathbf{t} : T$ . Induction over  $n$ .

*Basic cases.*  $n = 1$ .

1. 
$$\frac{\frac{\Gamma, x : U, y : V \vdash y : V}{\Gamma, y : V \vdash \lambda x y : U \rightarrow V} ax}{\Gamma, y : V \vdash \lambda x y : U \rightarrow V} \rightarrow I[U]$$
2. 
$$\frac{\Gamma, x : U \vdash x : U}{\Gamma \vdash \lambda x x : U \rightarrow U} ax \rightarrow I[U]$$
3. 
$$\frac{\Gamma, x : U \vdash \mathbf{0} : \bar{0}}{\Gamma \vdash \lambda x \mathbf{0} : U \rightarrow \bar{0}} ax_{\bar{0}} \rightarrow I[U]$$

*Inductive cases.*

1. 
$$\frac{\Gamma, x : U \vdash \mathbf{t} : A}{\Gamma \vdash \lambda x \mathbf{t} : U \rightarrow A} \rightarrow I[U]$$
 This is the trivial case.
2. 
$$\frac{\Gamma \vdash \lambda x \mathbf{t} : \forall X. T}{\Gamma \vdash \lambda x \mathbf{t} : T[V/X]} \forall E$$
 by the induction hypothesis  $\exists U, A$  s.t.  $\Gamma, x : U \vdash \mathbf{t} : A$  and  $U \rightarrow A \geq \forall X. T > T[V/X]$ .
3. 
$$\frac{\Gamma \vdash \lambda x \mathbf{t} : T}{\Gamma \vdash \lambda x \mathbf{t} : \forall X. T} \forall I$$
 by the induction hypothesis  $\exists U, A$  s.t.  $\Gamma, x : U \vdash \mathbf{t} : A$  and  $U \rightarrow A \geq T > \forall X. T$ .

**Lemma 8 (Generation lemma (sc)).** Let  $\alpha \neq 0$  and  $S_n = \Gamma \vdash \alpha. \mathbf{t} : \alpha. T$ . Then  $\exists m < n$  s.t.  $S_m = \Gamma \vdash \mathbf{t} : T$

PROOF. Induction over  $n$ .

*Basic case.*  $n = 1$ .

1. 
$$\frac{\Gamma, x : U \vdash x : U}{\Gamma, x : U \vdash \alpha. x : \alpha. U} ax$$
  
Then  $S_0 = \Gamma, x : U \vdash x : U$ .
2. 
$$\frac{\Gamma \vdash \mathbf{0} : \bar{0}}{\Gamma \vdash \alpha. \mathbf{0} : \alpha. \bar{0}} ax_{\bar{0}}$$
  
Then  $S_0 = \Gamma \vdash \mathbf{0} : \bar{0}$ .

*Inductive cases.* Looking at the last derivation rule

1. 
$$\frac{\Gamma \vdash \alpha. \mathbf{t} : \forall X. \alpha. B}{\Gamma \vdash \alpha. \mathbf{t} : \alpha. B[U/X]} \forall E$$
 As  $\forall X. \alpha. B \equiv \alpha. \forall X. B$ , by the induction hypothesis  $\exists m < n - 1$  s.t.  $S_m = \Gamma \vdash \mathbf{t} : \forall X. B$ , then by using  $\forall E$  rule,  $\Gamma \vdash \mathbf{t} : B[U/X]$  and notice that  $m < n - 1 \Rightarrow m + 1 < n$ .

2. 
$$\frac{\frac{\Gamma \vdash \alpha.t : \alpha.B}{\Gamma \vdash \alpha.t : \forall X.\alpha.B} \forall I}{\Gamma \vdash \alpha.t : \alpha.\forall X.B} \equiv$$
 by the induction hypothesis  $\exists m < n - 1$  s.t.  $S_m = \Gamma \vdash t : B$ , then by using  $\forall I$  rule,  $\Gamma \vdash t : \forall X.B$  and notice that  $m < n - 1 \Rightarrow m + 1 < n$ .
3. 
$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \alpha.t : \alpha.A} \alpha.I$$
 This is the trivial case.

**Lemma 9 (Generation lemma (sum)).** *Let  $S_n = \Gamma \vdash \mathbf{u} + \mathbf{v} : \alpha.A$ . Then  $\exists \delta, \gamma \in \mathcal{S}$  and  $r, s \in \mathbb{N}_0$  s.t.  $S_r = \Gamma \vdash \mathbf{u} : \delta.A$ ,  $S_s = \Gamma \vdash \mathbf{v} : \gamma.A$ ,  $\delta + \gamma = \alpha$  and  $\max(r, s) < n$ .*

PROOF. Induction over  $n$ .

*Basic cases.*  $n = 1$ . We enumerate the four possible ways of deriving  $\Gamma \vdash \mathbf{u} + \mathbf{v} : \alpha.A$  in a derivation tree of size 1.  $S_r$  and  $S_s$  turn out as sub-trees.

1. 
$$\frac{\frac{\frac{\Gamma, x : U, y : U \vdash x : 1.U}{\Gamma, x : U, y : U \vdash x + y : (1 + 1).U} ax \text{ and } \equiv}{\Gamma, x : U, y : U \vdash x + y : (1 + 1).U} +I}{\Gamma, x : U, y : U \vdash x + y : (1 + 1).U} +I$$
2. 
$$\frac{\frac{\frac{\Gamma, x : U \vdash x : 1.U}{\Gamma, x : U \vdash x + \mathbf{0} : (1 + 0).U} ax \text{ and } \equiv}{\Gamma, x : U \vdash x + \mathbf{0} : (1 + 0).U} +I}{\Gamma, x : U \vdash x + \mathbf{0} : (1 + 0).U} +I$$
3. 
$$\frac{\frac{\frac{\Gamma, y : U \vdash \mathbf{0} : 0.U}{\Gamma, y : U \vdash \mathbf{0} + y : (0 + 1).U} ax_{\overline{0}} \text{ and } \equiv}{\Gamma, y : U \vdash \mathbf{0} + y : (0 + 1).U} +I}{\Gamma, y : U \vdash \mathbf{0} + y : (0 + 1).U} +I$$
4. 
$$\frac{\frac{\frac{\Gamma \vdash \mathbf{0} : 0.U}{\Gamma \vdash \mathbf{0} + \mathbf{0} : (0 + 0).U} ax_{\overline{0}} \text{ and } \equiv}{\Gamma \vdash \mathbf{0} + \mathbf{0} : (0 + 0).U} +I}{\Gamma \vdash \mathbf{0} + \mathbf{0} : (0 + 0).U} +I$$

*Inductive cases.* We suppose that any derivation of size  $n - 1$  of  $\Gamma \vdash \mathbf{u} + \mathbf{v} : \alpha.A$ , has the property above. Looking at the last derivation rule, the possible cases are

1. 
$$\frac{\Gamma \vdash \mathbf{u} : \delta.A \quad \Gamma \vdash \mathbf{v} : (\alpha - \delta).A}{\Gamma \vdash \mathbf{u} + \mathbf{v} : \alpha.A} +I$$
 Then take  $\gamma = \alpha - \delta$ ,  $S_r = \Gamma \vdash \mathbf{u} : \delta.A$ ,  $S_s = \Gamma \vdash \mathbf{v} : (\alpha - \delta).A$  and notice that  $\max(r, s) < n$  and  $\delta + \alpha - \delta = \alpha$ .
2. 
$$\frac{\Gamma \vdash \mathbf{u} + \mathbf{v} : \forall X.\alpha.B}{\Gamma \vdash \mathbf{u} + \mathbf{v} : \alpha.B[U/X]} \forall E$$
 As  $\forall X.\alpha.B \equiv \alpha.\forall X.B$ , by the induction hypothesis  $\exists \delta, \gamma, r$  and  $s$  s.t.  $S_s = \Gamma \vdash \mathbf{u} : \delta.\forall X.B$ ,  $S_r = \Gamma \vdash \mathbf{v} : \gamma.\forall X.B$ ,  $\delta + \gamma = \alpha$  and  $\max(r, s) < n - 1$ . Then by using  $\forall E$  rule,  $\Gamma \vdash \mathbf{u} : \delta.B[U/X]$  and  $\Gamma \vdash \mathbf{v} : \gamma.B[U/X]$ . So,  $S_{r+1} = \Gamma \vdash \mathbf{u} : \delta.B[U/X]$ ,  $S_{s+1} = \Gamma \vdash \mathbf{v} : \gamma.B[U/X]$  and  $\max(r + 1, s + 1) = \max(r, s) + 1 < n$ .
3. 
$$\frac{\frac{\Gamma \vdash \mathbf{u} + \mathbf{v} : \alpha.B}{\Gamma \vdash \mathbf{u} + \mathbf{v} : \forall X.\alpha.B} \forall I}{\Gamma \vdash \mathbf{u} + \mathbf{v} : \alpha.\forall X.B} \equiv$$
 by the induction hypothesis  $\exists \delta, \gamma, r$  and  $s$  s.t.  $S_r = \Gamma \vdash \mathbf{u} : \delta.B$ ,  $S_s = \Gamma \vdash \mathbf{v} : \gamma.B$ ,  $\delta + \gamma = \alpha$  and  $\max(r, s) < n - 1$ . Then, by using  $\forall I$  rule,  $\Gamma \vdash \mathbf{u} : \forall X.\delta.B \equiv \delta.\forall X.B$  and  $\Gamma \vdash \mathbf{v} : \forall X.\gamma.B \equiv \gamma.\forall X.B$ . So,  $S_{r+1} = \Gamma \vdash \mathbf{u} : \delta.\forall X.B$ ,  $S_{s+1} = \Gamma \vdash \mathbf{v} : \gamma.\forall X.B$  and  $\max(r + 1, s + 1) = \max(r, s) + 1 < n$ .

The following lemma is quite standard in proofs of subject reduction for System  $F$ -like systems, and can be found in [6, 26]. It ensures that by substituting type variables for types or term variables for terms in an adequate manner, the type derived is still valid.

**Lemma 10 (Substitution lemma).**

1.  $\Gamma \vdash \mathbf{u}:T \Rightarrow \Gamma[U/X] \vdash \mathbf{u}:T[U/X]$ , with  $U \in \mathcal{U}$ .
2.  $\Gamma, x:U \vdash \mathbf{t}:B \wedge \Gamma \vdash \mathbf{b}:U \Rightarrow \Gamma \vdash \mathbf{t}[\mathbf{b}/x]:B$ , with  $U \in \mathcal{U}$ .

PROOF.

1. Induction on the derivation of  $\Gamma \vdash \mathbf{u}:T$ . See appendix Appendix E.
2. Induction on the derivation of  $\Gamma, x:U \vdash \mathbf{t}:B$ . See appendix Appendix F.

The following corollary allows the arrow to be split without needing to consider the order relation:

**Corollary 1 (of lemma 7).**  $\Gamma \vdash \lambda x \mathbf{t}:U \rightarrow T \Rightarrow \Gamma, x:U \vdash \mathbf{t}:T$

PROOF. Let  $\Gamma \vdash \lambda x \mathbf{t}:U \rightarrow T$ . By lemma 7,  $\exists V, R$  such that  $V \rightarrow R \geq U \rightarrow T$  and  $\Gamma, x:V \vdash \mathbf{t}:R$ , then by lemma 4,  $\exists \vec{W}, \vec{X}$  such that  $U \rightarrow T \equiv (V \rightarrow R)[\vec{W}/\vec{X}]$  and by lemma 10,  $(\Gamma, x)[\vec{W}/\vec{X}]:V[\vec{W}/\vec{X}] \vdash \mathbf{t}:R[\vec{W}/\vec{X}]$ , i.e.  $\Gamma[\vec{W}/\vec{X}], x:U \vdash \mathbf{t}:T$ .

Notice that if  $\Gamma[\vec{W}/\vec{X}] \equiv \Gamma$ , then we are finished. In the other case,  $\vec{X}$  appears free on  $\Gamma$ , however, to get  $U \rightarrow T$  from  $V \rightarrow R$  as a type for  $\lambda x \mathbf{t}$  by substitutions, we would need to use the rule  $\forall I$ , so  $\vec{X}$  cannot appear free in  $\Gamma$ , which constitutes a contradiction. So,  $\Gamma, x:U \vdash \mathbf{t}:T$ .

#### 4.1.3. Lemmas about the scalars

This section contains the lemmas which make statements about the relative behaviour of the scalars within terms and within types.

For example, scalars appearing in the terms must found themselves reflected within the types also. This is formalised in following lemma:

**Lemma 11 (Scaling unit).**  $\Gamma \vdash \alpha.\mathbf{t}:T \Rightarrow \exists U \in \mathcal{U}, \gamma \in \mathcal{S} \text{ s.t. } T \equiv \alpha.\gamma.U$

PROOF. Let  $S_n = \Gamma \vdash \alpha.\mathbf{t}:T$ . Induction over  $n$ .

*Basic cases.*  $n = 1$ .

1.  $\frac{\overline{\Gamma, x:U \vdash x:U}^{ax}}{\Gamma, x:U \vdash \alpha.x:\alpha.U} sI[\alpha]$  Notice that  $\forall U \in \mathcal{U}, U \equiv 1.U$ .
2.  $\frac{\overline{\Gamma \vdash \mathbf{0}:\bar{0}}^{ax\bar{0}}}{\Gamma \vdash \alpha.\mathbf{0}:\alpha.\bar{0}} sI[\alpha]$  Notice that  $\forall U \in \mathcal{U}, \bar{0} \equiv 0.U$ .

*Inductive cases.* The possible cases are

1.  $\frac{\Gamma \vdash \mathbf{t}:A}{\Gamma \vdash \alpha.\mathbf{t}:\alpha.A} sI[\alpha]$  By lemma 1,  $\exists U \in \mathcal{U}, \gamma \in \mathcal{S} \text{ s.t. } A \equiv \gamma.U$ . Then  $\alpha.A \equiv \alpha.\gamma.U$ .  
By lemma 1,  $\exists U \in \mathcal{U}, \delta \in \mathcal{S} \text{ s.t. } B \equiv \delta.U$ , then  $B[V/X] \equiv \delta.U[V/X]$  and also  $\forall X.B \equiv \forall X.\delta.U \equiv \delta.\forall X.U$ . In addition, by the induction hypothesis  $\exists U' \in \mathcal{U}, \gamma \in \mathcal{S} \text{ s.t. } \forall X.B \equiv \alpha.\gamma.U'$ . Summarising:  $\alpha.\gamma.U' \equiv \delta.\forall X.U$ . Then, by lemma 2,  $\delta = \alpha \times \gamma$ , so  $B[V/X] \equiv \alpha.\gamma.U[V/X]$ . In addition, by lemma 1,  $\exists U'' \in \mathcal{U}, \varsigma \in \mathcal{S} \text{ s.t. } U[A/X] \equiv \varsigma.U''$ . Then  $B[A/X] \equiv \alpha.\gamma.\varsigma.U'' \equiv \alpha.(\gamma \times \varsigma).U''$ .
2.  $\frac{\Gamma \vdash \alpha.\mathbf{t}:\forall X.B}{\Gamma \vdash \alpha.\mathbf{t}:B[V/X]} \forall E$
3.  $\frac{\Gamma \vdash \alpha.\mathbf{t}:B}{\Gamma \vdash \alpha.\mathbf{t}:\forall X.B} \forall I$  by the induction hypothesis  $\exists U \in \mathcal{U}, \gamma \in \mathcal{S} \text{ s.t. } B \equiv \alpha.\gamma.U$ , then  $\forall X.B \equiv \forall X.\alpha.\gamma.U \equiv \alpha.\gamma.\forall X.U$ .

A base term can always be given a unit type.

**Lemma 12 (Base terms in unit).** *Let  $\mathbf{b}$  be a base term. Then  $\Gamma \vdash \mathbf{b}:T \Rightarrow \exists U \in \mathcal{U} \text{ s.t. } T \equiv U$ .*

PROOF. Induction on the derivation of  $\Gamma \vdash \mathbf{b}:T$ . See appendix Appendix G.

By  $ax_{\bar{0}}$  is easy to see that  $\mathbf{0}$  has type  $\bar{0}$ , but also by using equivalences between types it is easy to see that  $\forall X.\bar{0}$  is equivalent to  $\bar{0}$  and any  $T \leq \bar{0}$  will also be equivalent to  $\bar{0}$ . Then:

**Lemma 13 (Type for 0).**  $\Gamma \vdash \mathbf{0}:T \Rightarrow T \equiv \bar{0}$

PROOF. Induction on the derivation of  $\Gamma \vdash \mathbf{0}:\bar{0}$ . See appendix Appendix H.

The following theorem is an important one. It says that our *scalar* type system is polymorphic only in the unit types but not in the general types in the sense that even if it is possible to derive two types for the same term, the outer left scalar (*i.e.* scalar in the head position) must remain the same. Its proof is not trivial and uses several of the previously defined lemmas.

**Theorem 2 (Uniqueness of scalars).** *Let  $U, V \in \mathcal{U}$ . Then*

$$\left. \begin{array}{l} \Gamma \vdash \mathbf{t}:\alpha.U \\ \Gamma \vdash \mathbf{t}:\beta.V \end{array} \right\} \Rightarrow \alpha = \beta$$

PROOF. Structural induction over  $\mathbf{t}$ .

*Basic cases.*

1.  $\mathbf{t} = \mathbf{0}$ . Then by lemma 13,  $\alpha = \beta = 0$ .
2.  $\mathbf{t} = x$ . Then by lemma 12,  $\alpha = \beta = 1$ .
3.  $\mathbf{t} = \lambda x \mathbf{t}'$ . Then by lemma 12,  $\alpha = \beta = 1$ .

*Inductive cases.*

1.  $\mathbf{t} = \gamma.\mathbf{t}'$ . Then by lemma 11,  $\exists \sigma, \delta, U', V'$ , s.t.  $\alpha.U \equiv \gamma.\sigma.U'$  and  $\beta.V \equiv \gamma.\delta.V'$ . Then by lemma 8,  $\Gamma \vdash \mathbf{t}':\sigma.U'$  and  $\Gamma \vdash \mathbf{t}':\delta.V'$ , so by the induction hypothesis  $\sigma = \delta$ . In addition, by lemma 2,  $\alpha = \gamma \times \sigma$  and  $\beta = \gamma \times \delta$ , so  $\alpha = \gamma \times \sigma = \gamma \times \delta = \beta$ .
2.  $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$ . Then by lemma 9,  $\exists \gamma_1, \gamma_2, \delta_1, \delta_2$  s.t.

$$\left\{ \begin{array}{l} \Gamma \vdash \mathbf{t}_1:\gamma_1.U \\ \Gamma \vdash \mathbf{t}_2:\gamma_2.U \\ \gamma_1 + \gamma_2 = \alpha \end{array} \right\} \quad \left\{ \begin{array}{l} \Gamma \vdash \mathbf{t}_1:\delta_1.V \\ \Gamma \vdash \mathbf{t}_2:\delta_2.V \\ \delta_1 + \delta_2 = \beta \end{array} \right\}$$

Then by the induction hypothesis  $\gamma_1 = \delta_1$  and  $\gamma_2 = \delta_2$ , so  $\alpha = \gamma_1 + \gamma_2 = \delta_1 + \delta_2 = \beta$ .

3.  $\mathbf{t} = (\mathbf{t}_1 \ \mathbf{t}_2)$ . Then by lemma 6,  $\exists \gamma_1, \gamma_2, \delta_1, \delta_2, W, W', U' \geq U, V' \geq V$  s.t.

$$\left\{ \begin{array}{l} \Gamma \vdash \mathbf{t}_1:\gamma_1.W \rightarrow U' \\ \Gamma \vdash \mathbf{t}_2:\gamma_2.W \\ \gamma_1 \times \gamma_2 = \alpha \end{array} \right\} \quad \left\{ \begin{array}{l} \Gamma \vdash \mathbf{t}_1:\delta_1.W' \rightarrow V' \\ \Gamma \vdash \mathbf{t}_2:\delta_2.W' \\ \delta_1 \times \delta_2 = \beta \end{array} \right\}$$

Then by the induction hypothesis  $\gamma_1 = \delta_1$  and  $\gamma_2 = \delta_2$ , so  $\alpha = \gamma_1 \times \gamma_2 = \delta_1 \times \delta_2 = \beta$ .

From this theorem, the uniqueness of  $\bar{0}$  comes out, in the sense that no term can have type  $\bar{0}$  and some other type  $T$  which is not equivalent to  $\bar{0}$ .

**Corollary 2 (Uniqueness of  $\bar{0}$ ).**  $\Gamma \vdash \mathbf{t}:\bar{0} \Rightarrow \forall T \neq \bar{0}, \Gamma \not\vdash \mathbf{t}:T$ .

PROOF. See appendix Appendix I

As  $\mathbf{0}$  has type  $\bar{0}$  which is equivalent to  $0.U$  for any  $U$ ,  $\mathbf{0}$  can still be given as an argument for an application, or even be applied to another term. In either case the result will be a term of type  $\bar{0}$ :

**Lemma 14 (Linearity of  $\mathbf{0}$ ).**

1.  $\Gamma \vdash \mathbf{0} \mathbf{u}: T \Rightarrow T \equiv \bar{0}$ .
2.  $\Gamma \vdash \mathbf{u} \mathbf{0}: T \Rightarrow T \equiv \bar{0}$ .

PROOF.

1. Let  $\Gamma \vdash \mathbf{0} \mathbf{u}: T$  and using lemma 1, let  $T \equiv \gamma.U$ . Then by lemma 6,  $\exists \alpha, \beta, U'$  and  $B \geq U$  s.t.

$$\begin{cases} \gamma = \alpha \times \beta \\ \Gamma \vdash \mathbf{0}: \beta.U' \rightarrow B \\ \Gamma \vdash \mathbf{u}: \alpha.U' \end{cases}$$

Hence, by corollary 2,  $\beta.U' \rightarrow B \equiv \bar{0} \equiv 0.U$ , so by lemma 2,  $\beta = 0$ , then  $\gamma = \alpha \times 0 = 0$ , so  $T \equiv \gamma.U \equiv \bar{0}$ .

2. Analogous to 1.

4.1.4. *Subject reduction cases.*

The following three lemmas are in fact cases of subject reduction, however, they will also be necessary as lemmas in subsequent proofs.

**Lemma 15 (Product).**  $\Gamma \vdash \alpha.(\beta.\mathbf{u}): T \Rightarrow \Gamma \vdash (\alpha \times \beta).\mathbf{u}: T$ .

PROOF. By lemma 11,  $\exists U \in \mathcal{U}, \gamma \in \mathcal{S}$  s.t.  $T \equiv \alpha.\gamma.U$ . Then by lemma 8,  $\Gamma \vdash \beta.\mathbf{u}: \gamma.U$ . Then by lemma 11 again,  $\exists U' \in \mathcal{U}, \gamma' \in \mathcal{S}$  s.t.  $\gamma.U \equiv \beta.\gamma'.U'$ . Then by lemma 8,  $\Gamma \vdash \mathbf{u}: \gamma'.U'$ , so

$$\frac{\Gamma \vdash \mathbf{u}: \gamma'.U'}{\Gamma \vdash (\alpha \times \beta).\mathbf{u}: (\alpha \times \beta).\gamma'.U'} sI[\alpha \times \beta]$$

Notice that  $(\alpha \times \beta).\gamma'.U' \equiv \alpha.\beta.\gamma'.U' \equiv \alpha.\gamma.U \equiv T$ .

**Lemma 16 (Distributivity).**  $\Gamma \vdash \alpha.(\mathbf{u} + \mathbf{v}): T \Rightarrow \Gamma \vdash \alpha.\mathbf{u} + \alpha.\mathbf{v}: T$

PROOF. Let  $\Gamma \vdash \alpha.(\mathbf{u} + \mathbf{v}): T$ . Then by lemma 11,  $\exists \alpha$  s.t.  $T \equiv \alpha.A$ , so by lemma 8,  $\Gamma \vdash \mathbf{u} + \mathbf{v}: A$

Using the fact that  $A \equiv 1.A$ , by lemma 9,  $\Gamma \vdash \mathbf{u}: \delta.A$  and  $\Gamma \vdash \mathbf{v}: (1 - \delta).A$ . Then

$$\frac{\frac{\Gamma \vdash \mathbf{u}: \delta.A}{\Gamma \vdash \alpha.\mathbf{u}: (\alpha \times \delta).A} sI[\alpha] \text{ and } \frac{\frac{\mathbf{v}: (1 - \delta).A}{\Gamma \vdash \alpha.\mathbf{v}: (\alpha \times (1 - \delta)).A} sI[\alpha] \text{ and } \equiv}{\Gamma \vdash \alpha.\mathbf{u} + \alpha.\mathbf{v}: (\alpha \times \delta + \alpha \times (1 - \delta)).A} +I$$

Notice that  $(\alpha \times \delta + \alpha \times (1 - \delta)).A = \alpha.A \equiv T$ .

**Lemma 17 (Factorisation).**  $\Gamma \vdash \alpha.\mathbf{u} + \beta.\mathbf{u}: T \Rightarrow \Gamma \vdash (\alpha + \beta).\mathbf{u}: T$

PROOF. Let  $\Gamma \vdash \alpha.\mathbf{u} + \beta.\mathbf{u}: T$ . By lemma 9,  $\exists \delta, \gamma \in \mathcal{S}$  s.t.

$$\begin{cases} \Gamma \vdash \alpha.\mathbf{u}: \delta.T \\ \Gamma \vdash \beta.\mathbf{u}: \gamma.T \\ \delta + \gamma = 1 \end{cases}$$

In addition, by lemma 1,  $\exists U \in \mathcal{U}$  and  $\sigma \in \mathcal{S}$  s.t.  $T \equiv \sigma.U$ . Then

$$\begin{cases} \Gamma \vdash \alpha.\mathbf{u}: \delta.\sigma.U \\ \Gamma \vdash \beta.\mathbf{u}: \gamma.\sigma.U \\ \delta + \gamma = 1 \end{cases}$$

Then by lemma 11,  $\exists \phi, \varphi \in \mathcal{S}$  and  $U', U'' \in \mathcal{U}$  s.t.

$$\begin{cases} \delta \cdot \sigma \cdot U \equiv \alpha \cdot \phi \cdot U' \\ \gamma \cdot \sigma \cdot U \equiv \beta \cdot \varphi \cdot U'' \end{cases}$$

So, by lemma 2,

$$\begin{cases} \delta \times \sigma = \alpha \times \phi \\ \gamma \times \sigma = \beta \times \varphi \end{cases}$$

Cases:

1. Case  $\sigma = 0$ . Then  $T \equiv \bar{0}$ , so  $\Gamma \vdash \alpha \cdot \mathbf{u} : \alpha \cdot \bar{0}$ , so by lemma 8,  $\Gamma \vdash \mathbf{u} : \bar{0}$ . Then

$$\frac{\Gamma \vdash \mathbf{u} : \bar{0}}{\Gamma \vdash (\alpha + \beta) \cdot \mathbf{u} : \bar{0}} sI[\alpha + \beta] \text{ and } \equiv$$

2. Case  $\sigma \neq 0, \delta = 0$ . Then  $\gamma = 1$ , so  $\Gamma \vdash \beta \cdot \mathbf{u} : T \equiv \beta \cdot \varphi \cdot U''$ , then by lemma 8,  $\Gamma \vdash \mathbf{u} : \varphi \cdot U''$ , so

$$\frac{\Gamma \vdash \mathbf{u} : \varphi \cdot U''}{\Gamma \vdash (\alpha + \beta) \cdot \mathbf{u} : ((\alpha + \beta) \times \varphi) \cdot U''} sI[\alpha + \beta] \text{ and } \equiv$$

As  $\delta = 0$ , the possible cases are

- $\alpha = 0$ , so  $((\alpha + \beta) \times \varphi) \cdot U'' \equiv (\beta \times \varphi) \cdot U'' \equiv \sigma \cdot U \equiv T$ .

- $\alpha \neq 0$ , then  $\Gamma \vdash \alpha \cdot \mathbf{u} : \bar{0} \equiv \alpha \cdot \bar{0}$ , then by lemma 8,  $\Gamma \vdash \mathbf{u} : \bar{0}$ .

In addition, as  $\Gamma \vdash \beta \cdot \mathbf{u} : \beta \cdot \varphi \cdot U''$ , by lemma 8,  $\Gamma \vdash \mathbf{u} : \varphi \cdot U''$ , then by corollary 2,  $\varphi \cdot U'' \equiv \bar{0}$ , so  $\varphi = 0$ , and then  $\gamma = 0$ , so  $\delta = 1$ , which is a contradiction.

3. Case  $\sigma \neq 0, \gamma = 0$ . Analogous 2.
4. Case  $\alpha, \beta, \phi, \varphi$  not 0. Then by lemma 2,  $U \equiv U' \equiv U''$

Then

$$\begin{cases} \Gamma \vdash \alpha \cdot \mathbf{u} : \alpha \cdot \phi \cdot U \\ \Gamma \vdash \beta \cdot \mathbf{u} : \beta \cdot \varphi \cdot U \\ \delta + \gamma = 1 \end{cases}$$

Hence by lemma 8,  $\Gamma \vdash \mathbf{u} : \phi \cdot U$  and  $\Gamma \vdash \mathbf{u} : \varphi \cdot U$

Then by theorem 2,  $\phi = \varphi$  and then

$$\frac{\Gamma \vdash \mathbf{u} : \phi \cdot U}{\Gamma \vdash (\alpha + \beta) \cdot \mathbf{u} : (\alpha + \beta) \cdot \phi \cdot U} sI[\alpha + \beta]$$

Notice that  $(\alpha + \beta) \cdot \phi \cdot U \equiv ((\alpha + \beta) \times \phi) \cdot U = (\alpha \times \phi + \beta \times \varphi) \cdot U = (\delta \times \sigma + \gamma \times \sigma) \cdot U = ((\delta + \gamma) \times \sigma) \cdot U = (1 \times \sigma) \cdot U = \sigma \cdot U \equiv T$ .

#### 4.2. Subject reduction proof

Now we are able to prove subject reduction (Theorem 1).

PROOF. We proceed by checking that every reduction rule preserves the type.

#### Group E

**rule  $\mathbf{u} + \mathbf{0} \rightarrow \mathbf{u}$ .** Let  $\Gamma \vdash \mathbf{u} + \mathbf{0} : T$ . Then by lemma 9,  $\exists \alpha, \beta \in \mathcal{S}$  s.t.  $\alpha + \beta = 1$ ,  $\Gamma \vdash \mathbf{u} : \alpha \cdot T$  and  $\Gamma \vdash \mathbf{0} : \beta \cdot T$ . Then, by lemma 13,  $\beta \cdot T \equiv \bar{0}$ , so  $T \equiv \bar{0}$ , and then  $\alpha \cdot T \equiv \bar{0} \equiv T$ , or  $\beta = 0$ , so  $\alpha = 1$ .

**rule  $\mathbf{0} \cdot \mathbf{u} \rightarrow \mathbf{0}$ .** Let  $\Gamma \vdash \mathbf{0} \cdot \mathbf{u} : T$ , then by lemma 11,  $\exists A$  s.t.  $T \equiv \mathbf{0} \cdot A \equiv \bar{0}$ , and by rule  $ax_{\bar{0}}$ ,  $\Gamma \vdash \mathbf{0} : \bar{0}$ .

**rule  $\mathbf{1} \cdot \mathbf{u} \rightarrow \mathbf{u}$ .** Let  $\Gamma \vdash \mathbf{1} \cdot \mathbf{u} : T \equiv \mathbf{1} \cdot T$ . Then by lemma 8,  $\Gamma \vdash \mathbf{u} : T$ .

**rule  $\alpha.\mathbf{0} \rightarrow \mathbf{0}$ .** Let  $\Gamma \vdash \alpha.\mathbf{0}:T$ , then by lemma 11,  $\exists A$  s.t.  $T \equiv \alpha.A$ , then by lemma 8,  $\Gamma \vdash \mathbf{0}:A$ . So, by rule  $ax_{\overline{0}}$ ,  $A \equiv \overline{0}$  and so  $T \equiv \alpha.A \equiv \overline{0}$ .

**rule  $\alpha.(\beta.\mathbf{u}) \rightarrow (\alpha \times \beta).\mathbf{u}$ .** True by lemma 15

**rule  $\alpha.(\mathbf{u} + \mathbf{v}) \rightarrow \alpha.\mathbf{u} + \alpha.\mathbf{v}$ .** True by lemma 16.

### Group F

**rule  $\alpha.\mathbf{u} + \beta.\mathbf{u} \rightarrow (\alpha + \beta).\mathbf{u}$ .** True by lemma 17.

**rule  $\alpha.\mathbf{u} + \mathbf{u} \rightarrow (\alpha + 1).\mathbf{u}$ .** Let  $\Gamma \vdash \alpha.\mathbf{u} + \mathbf{u}:T$ . Then

$$\frac{\Gamma \vdash \alpha.\mathbf{u} + \mathbf{u}:T}{\Gamma \vdash 1.(\alpha.\mathbf{u} + \mathbf{u}):1.T} sI[1]$$

Then by lemma 16,  $\Gamma \vdash 1.\alpha.\mathbf{u} + 1.\mathbf{u}:1.T$ . Then by lemma 9

$$\left\{ \begin{array}{l} \Gamma \vdash 1.\alpha.\mathbf{u}:\gamma.T \\ \Gamma \vdash 1.\mathbf{u}:\delta.T \\ \gamma + \delta = 1 \end{array} \right.$$

Then by lemma 15,  $\Gamma \vdash \alpha.\mathbf{u}:\gamma.T$ , so

$$\frac{\Gamma \vdash \alpha.\mathbf{u}:\gamma.T \quad \Gamma \vdash 1.\mathbf{u}:\delta.T}{\Gamma \vdash \alpha.\mathbf{u} + 1.\mathbf{u}:1.T} +I$$

Then by lemma 17,  $\Gamma \vdash (\alpha + 1).\mathbf{u}:1.T \equiv T$ .

**rule  $\mathbf{u} + \mathbf{u} \rightarrow (1 + 1).\mathbf{u}$ .** Let  $\Gamma \vdash \mathbf{u} + \mathbf{u}:T$ . Then by rule  $sI[1]$ ,  $\Gamma \vdash 1.(\mathbf{u} + \mathbf{u}):1.T$ . So by lemma 16,  $\Gamma \vdash 1.\mathbf{u} + 1.\mathbf{u}:1.T$ . Then by lemma 17,  $\Gamma \vdash (1 + 1).\mathbf{u}:1.T \equiv T$ .

### Group A

**rule  $(\mathbf{u} + \mathbf{v}) \mathbf{w} \rightarrow (\mathbf{u} \mathbf{w}) + (\mathbf{v} \mathbf{w})$ .** Let  $\Gamma \vdash (\mathbf{u} + \mathbf{v}) \mathbf{w}:T \equiv 1.T$ . Then, by lemma 6,  $\exists \alpha, \beta, U$  and  $T' \geq T$  s.t.

$$\left\{ \begin{array}{l} \alpha \times \beta = 1 \\ \Gamma \vdash \mathbf{w}:\alpha.U \\ \Gamma \vdash \mathbf{u} + \mathbf{v}:\beta.U \rightarrow T' \equiv 1.\beta.U \rightarrow T' \end{array} \right.$$

Then by lemma 9,  $\exists \delta$  s.t.

$$\left\{ \begin{array}{l} \Gamma \vdash \mathbf{u}:\delta.\beta.U \rightarrow T' \equiv (\delta \times \beta).U \rightarrow T' \\ \Gamma \vdash \mathbf{v}:(1 - \delta).\beta.U \rightarrow T' \equiv ((1 - \delta) \times \beta).U \rightarrow T' \end{array} \right.$$

Then

$$\frac{\Gamma \vdash \mathbf{u}:(\delta \times \beta).U \rightarrow T' \quad \Gamma \vdash \mathbf{w}:\alpha.U}{\Gamma \vdash (\mathbf{u} \mathbf{w}):(\delta \times \beta \times \alpha).T'} \rightarrow E$$

and  $(\delta \times \beta \times \alpha).T' = (\delta \times 1).T' = \delta.T'$ , then by lemmas 3 and 5  $\Gamma \vdash (\mathbf{u} \mathbf{w}):\delta.T$ .

In addition

$$\frac{\Gamma \vdash \mathbf{v}:(1 - \delta) \times \beta.(U \rightarrow T') \quad \Gamma \vdash \mathbf{w}:\alpha.U}{\Gamma \vdash (\mathbf{v} \mathbf{w}):((1 - \delta) \times \beta \times \alpha).T'} \rightarrow E$$

and  $((1 - \delta) \times \beta \times \alpha).T' = ((1 - \delta) \times 1).T' = (1 - \delta).T'$ , then by lemmas 3 and 5  $\Gamma \vdash (\mathbf{v} \mathbf{w}):(1 - \delta).T$ .

So

$$\frac{\Gamma \vdash (\mathbf{u} \mathbf{w}):\delta.T \quad \Gamma \vdash (\mathbf{v} \mathbf{w}):(1 - \delta).T}{\Gamma \vdash (\mathbf{u} \mathbf{w}) + (\mathbf{v} \mathbf{w}):T} +I \text{ and } \equiv$$

**rule  $\mathbf{w} (\mathbf{u} + \mathbf{v}) \rightarrow (\mathbf{w} \mathbf{u}) + (\mathbf{w} \mathbf{v})$ .** Analogous to the previous case.

**rule  $(\alpha.\mathbf{u}) \mathbf{v} \rightarrow \alpha.(\mathbf{u} \mathbf{v})$ .** Let  $\Gamma \vdash (\alpha.\mathbf{u}) \mathbf{v} : T \equiv 1.T$ . Then by lemma 6,  $\exists \gamma, \beta, U$  and  $T' \geq T$  s.t.

$$\begin{cases} \gamma \times \beta = 1 \\ \Gamma \vdash \mathbf{v} : \gamma.U \\ \Gamma \vdash \alpha.\mathbf{u} : \beta.U \rightarrow T' \end{cases}$$

and by lemma 11,  $\beta.U \rightarrow T' \equiv \alpha.\delta.U'$  then by lemma 2,  $U \rightarrow T' \equiv U'$  and  $\beta = \alpha \times \delta$  (notice that  $\beta \neq 0$  because  $\gamma \times \beta = 1$ ). So by lemma 8,  $\Gamma \vdash \mathbf{u} : \delta.(U \rightarrow T')$ , so

$$\frac{\Gamma \vdash \mathbf{u} : \delta.(U \rightarrow T') \quad \Gamma \vdash \mathbf{v} : \gamma.U}{\frac{\mathbf{u} \mathbf{v} : \delta.\gamma.T'}{\Gamma \vdash \alpha.(\mathbf{u} \mathbf{v}) : \alpha.\delta.\gamma.T'} \rightarrow E} sI[\alpha]$$

However,  $\alpha.\delta.\gamma.T' \equiv (\alpha \times \delta \times \gamma).T' = (\beta \times \gamma).T' = 1.T'$ , so by lemma 5,  $\Gamma \vdash \alpha.(\mathbf{u} \mathbf{v}) : 1.T \equiv T$ .

**rule  $\mathbf{v} (\alpha.\mathbf{u}) \rightarrow \alpha.(\mathbf{v} \mathbf{u})$ .** Analogous to the previous case.

**rule  $\mathbf{0} \mathbf{u} \rightarrow \mathbf{0}$ .** True by lemma 14, and rule  $ax_{\overline{0}}$ .

**rule  $\mathbf{u} \mathbf{0} \rightarrow \mathbf{0}$ .** True by lemma 14, and rule  $ax_{\overline{0}}$ .

### Group B

**rule  $(\lambda x \mathbf{t}) \mathbf{b} \rightarrow \mathbf{t}[\mathbf{b}/x]$ .** Let  $\Gamma \vdash (\lambda x \mathbf{t}) \mathbf{b} : T$ , then by lemma 6,  $\exists \alpha, \beta, U, T' \geq T$  s.t.

$$\begin{cases} 1 = \alpha \times \beta \\ \Gamma \vdash \lambda x \mathbf{t} : \beta.U \rightarrow T' \\ \Gamma \vdash \mathbf{b} : \alpha.U \end{cases}$$

As  $\mathbf{b}$  is a base term, then by lemma 12,  $\alpha = 1$  and so  $\beta = 1$ . Then by corollary 1,  $\Gamma, x : U \vdash \mathbf{t} : T'$ , so by lemma 10,  $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T'$ . Then by lemma 5,  $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T$ .

### AC equivalences

**Commutativity.** Let  $\Gamma \vdash \mathbf{u} + \mathbf{v} : T$ . Then, by lemma 9,  $\exists \delta \exists \gamma$  s.t.

$$\begin{cases} \Gamma \vdash \mathbf{u} : \delta.T \\ \Gamma \vdash \mathbf{v} : \gamma.T \\ \delta + \gamma = 1 \end{cases}$$

Then

$$\frac{\Gamma \vdash \mathbf{v} : \gamma.T \quad \Gamma \vdash \mathbf{u} : \delta.T}{\Gamma \vdash \mathbf{v} + \mathbf{u} : T} +I$$

**Associativity.** Let  $\Gamma \vdash (\mathbf{u} + \mathbf{v}) + \mathbf{w} : T$ . Then, by lemma 9,  $\exists \delta$  and  $\gamma$  s.t.

$$\begin{cases} \Gamma \vdash \mathbf{u} + \mathbf{v} : \delta.T \\ \Gamma \vdash \mathbf{w} : \gamma.T \\ \delta + \gamma = 1 \end{cases}$$

Then, by lemma 9, again  $\exists \delta'$  and  $\gamma'$  s.t.  $\Gamma \vdash \mathbf{u} : \delta'.T$ ,  $\Gamma \vdash \mathbf{v} : \gamma'.T$  and  $\delta' + \gamma' = \delta$ . Then

$$\frac{\Gamma \vdash \mathbf{u} : \delta'.T \quad \frac{\Gamma \vdash \mathbf{v} : \gamma'.T \quad \Gamma \vdash \mathbf{w} : \gamma.T}{\Gamma \vdash \mathbf{v} + \mathbf{w} : (\gamma' + \gamma).T} +I}{\Gamma \vdash \mathbf{u} + (\mathbf{v} + \mathbf{w}) : (\delta' + \gamma' + \gamma).T \equiv T} +I$$



## 5. Strong normalisation

The *scalar* type system will now be proved to have the strong normalisation property. In order to show this we first set up another type system, which simply ‘forgets’ the scalars. Hence this simpler type system is just a System  $F$  for *Lineal*, which we call  $\lambda 2^{la}$  (definition 5). In the literature surrounding not *Lineal* but its cousin, the algebraic  $\lambda$ -calculus, one finds such a System  $F$  in [15], which extends the simply typed algebraic  $\lambda$ -calculus of [40] – our  $\lambda 2^{la}$  is very similar. Secondly we prove strong normalisation for it (theorem 4). Thirdly we show that every term which has a type in *scalar* has a type in  $\lambda 2^{la}$  (lemma 19), which entails strong normalisation in *scalar* (theorem 5).

This strong normalisation proof constitutes the second main technical contribution of the paper. In case the reader is not interested by the technical details, he may read the strong normalisation theorem (theorem 5) and skip the remaining of this section continuing directly in section 6

In this section we use  $\Gamma \Vdash \mathbf{t} : T$  to say that it is possible to derive the type  $T \in \mathbb{T}(\lambda 2^{la})$  for the term  $\mathbf{t}$  in the context of  $\Gamma$  by using the typing rules from  $\lambda 2^{la}$ . We just use  $\vdash$  for *scalar*. In addition, we use  $Name^\triangleleft$  to distinguish the names of the typing rules in  $\lambda 2^{la}$ .

**Definition 5.** The typing rules of  $\lambda 2^{la}$  are the same as System  $F$  plus the following rules:

$$\frac{}{\Gamma \Vdash \mathbf{0} : A} ax_0^\triangleleft \quad \frac{\Gamma \Vdash \mathbf{u} : A \quad \Gamma \Vdash \mathbf{v} : A}{\Gamma \Vdash \mathbf{u} + \mathbf{v} : A} +I^\triangleleft \quad \frac{\Gamma \Vdash \mathbf{t} : A}{\Gamma \Vdash \alpha.\mathbf{t} : A} \alpha I^\triangleleft$$

In order to prove strong normalisation we extend the proof for  $\lambda 2$ . The standard method was invented by Tait [35] for simply typed  $\lambda$ -calculus and generalized to System  $F$  by Girard [18]. Our presentation follows [6, sec 4.3]. The following definitions are taken from this reference – with slight modifications to handle the extra  $\lambda 2^{la}$  rules.

The strong normalisation property entails that every term is strongly normalising, so first we define the set of strongly normalising terms.

**Definition 6.**  $SN = \{\mathbf{t} \in \Lambda \mid \mathbf{t} \text{ is strongly normalising}\}$ .

The notion of closure is often captured by the notion of saturated set:

**Definition 7.**

1. A subset  $X \subseteq SN$  is called *saturated* if
  - (a)  $\forall n \geq 0, x \mathbf{t}_1 \dots \mathbf{t}_n \in X$  where  $\mathbf{t}_i \in SN$  and  $x$  is any term variable;
  - (b)  $\forall n \geq 0, \mathbf{v}[\mathbf{b}/x] \mathbf{t}_1 \dots \mathbf{t}_n \in X \Rightarrow (\lambda x \mathbf{v}) \mathbf{b} \mathbf{t}_1 \dots \mathbf{t}_n \in X$ ;
  - (c)  $\mathbf{t}, \mathbf{u} \in X \Rightarrow \mathbf{t} + \mathbf{u} \in X$ ;
  - (d)  $\forall \alpha \in \mathcal{S}, \mathbf{t} \in X \Leftrightarrow \alpha.\mathbf{t} \in X$ ;
  - (e)  $\forall i \in I, \mathbf{u}_i \mathbf{w}_1 \dots \mathbf{w}_n \in X \Rightarrow (\sum_{i \in I} \mathbf{u}_i) \mathbf{w}_1 \dots \mathbf{w}_n \in X$ ;
  - (f)  $\forall i \in I, \mathbf{u} \mathbf{v}_i \mathbf{w}_1 \dots \mathbf{w}_n \in X \Rightarrow \mathbf{u} (\sum_{i \in I} \mathbf{v}_i) \mathbf{w}_1 \dots \mathbf{w}_n \in X$ ;
  - (g)  $\alpha.(\mathbf{t}_1 \dots \mathbf{t}_n) \in X \Leftrightarrow \mathbf{t}_1 \dots \alpha.\mathbf{t}_k \dots \mathbf{t}_n \in X$  ( $1 \leq k \leq n$ );
  - (h)  $\mathbf{0} \in X$ ;
  - (i)  $\forall \vec{\mathbf{t}} \in SN, (\mathbf{0} \vec{\mathbf{t}}) \in X$ ;
  - (j)  $\forall \mathbf{t}, \vec{\mathbf{u}} \in SN, (\mathbf{t} \mathbf{0}) \vec{\mathbf{u}} \in X$ .
2.  $SAT = \{X \subseteq \Lambda \mid X \text{ is saturated}\}$

The basic idea is to prove that types correspond to saturated sets. In order to achieve this, we define a valuation from types to  $SAT$  (in fact, from type variables to  $SAT$  and then, we define a set in  $SAT$  by using such a valuation).

**Definition 8.**

1. A *valuation* in  $SAT$  is a map  $\xi : \mathbb{V} \rightarrow SAT$ , where  $\mathbb{V}$  is the set of type variables.

2. Given a valuation  $\xi$  in  $SAT$  one defines for every  $T \in \mathbb{T}(\lambda 2^{la})$  a set  $\llbracket T \rrbracket_\xi \subseteq \Lambda$  as follows:

$$\begin{aligned} \llbracket X \rrbracket_\xi &= \xi(X), \text{ where } X \in \mathbb{V} \\ \llbracket A \rightarrow B \rrbracket_\xi &= \llbracket A \rrbracket_\xi \rightarrow \llbracket B \rrbracket_\xi \\ \llbracket \forall X. T \rrbracket_\xi &= \bigcap_{Y \in SAT} \llbracket T \rrbracket_{\xi(X:=Y)} \end{aligned}$$

**Lemma 18.**

1.  $SN \in SAT$ ,
2.  $U, T \in SAT \Rightarrow U \rightarrow T \in SAT$ ,
3. Let  $\{A_i\}_{i \in I}$  be a collection of members of  $SAT$ ,  $\bigcap_{i \in I} A_i \in SAT$ ,
4. Given a valuation  $\xi$  in  $SAT$  and a  $T$  in  $\mathbb{T}(\lambda 2^{la})$ , then  $\llbracket T \rrbracket_\xi \in SAT$ .

PROOF. See appendices Appendix J, Appendix K, Appendix L and Appendix M

Just like in definition 8, we define another valuation, this time from term variables to base terms. We use it to check what happens when we change every free variable of a term for any other base term. The basic idea is the following: we define  $\rho, \xi \models \mathbf{t}:T$  to be the property of changing every free term variable in  $\mathbf{t}$  for another term (a base term, as term variables only run over base terms) and still having the resulting term in the set  $\llbracket T \rrbracket_\xi$  for any valuation  $\xi$ . So, we define  $\Gamma \models \mathbf{t}:T$  to be the same property, when the property holds for every pair in  $\Gamma$  and for every valuations  $\rho$  and  $\xi$ .

This is formalised in the following definition (definition 9) and with this definition, we prove that if a term has a type in a valid context, then the property above holds (theorem 3), which will yield the strong normalisation theorem (theorem 4) via the concept of *saturated* set (because *saturated* sets are subsets of  $SN$ ).

**Definition 9.**

- A *valuation* in  $\Lambda$  is a map  $\rho: V \rightarrow \Lambda_b$ , where  $V$  is the set of term variables and  $\Lambda_b = \{\mathbf{b} \in \Lambda \mid \mathbf{b} \text{ is a base term}\}$ .
- Let  $\rho$  be a valuation in  $\Lambda$ . Then  $\llbracket \mathbf{t} \rrbracket_\rho = \mathbf{t}[x_1 := \rho(x_1), \dots, x_n := \rho(x_n)]$ , where  $\vec{x} = x_1, \dots, x_n$  is the set of free variables in  $\mathbf{t}$ .
- Let  $\rho$  be a valuation in  $\Lambda$  and  $\xi$  a valuation in  $SAT$ . Then
  - $\rho, \xi$  *satisfies*  $\mathbf{t}:T$ , notation  $\rho, \xi \models \mathbf{t}:T \Leftrightarrow \llbracket \mathbf{t} \rrbracket_\rho \in \llbracket T \rrbracket_\xi$ .
  - $\rho, \xi \models \Gamma \Leftrightarrow \rho, \xi \models x:T$  for all  $x:T$  in  $\Gamma$
  - $\Gamma \models \mathbf{t}:T \Leftrightarrow \forall \rho, \xi [\rho, \xi \models \Gamma \Rightarrow \rho, \xi \models \mathbf{t}:T]$ .

**Theorem 3 (Soundness).**  $\Gamma \vdash \mathbf{t}:T \Rightarrow \Gamma \models \mathbf{t}:T$ .

PROOF. We proceed by induction on the derivation of  $\Gamma \vdash \mathbf{t}:T$ .

*Basic cases*

1.  $\frac{}{\Gamma, x:A \vdash x:A} ax^\triangleleft$  Notice that if  $\rho, \xi \models \Gamma, x:A$ , then by definition  $\rho, \xi \models x:A$ .
2.  $\frac{}{\Gamma \vdash \mathbf{0}:A} ax_0^\triangleleft$  Then  $\forall \xi \forall \rho$ , by the saturation of  $\llbracket A \rrbracket_\xi$  one has  $\mathbf{0} \in \llbracket A \rrbracket_\xi$ . As  $\llbracket \mathbf{0} \rrbracket_\rho = \mathbf{0}$ , then  $\rho, \xi \models \mathbf{0}:A$ , and so  $\forall \Gamma, \Gamma \models \mathbf{0}:A$ .

*Inductive cases*

- by the induction hypothesis,  $\Gamma \models \mathbf{u} : A \rightarrow B$  and  $\Gamma \models \mathbf{v} : A$ . Assume  $\rho, \xi \models \Gamma$  in order to show  $\rho, \xi \models \mathbf{u} \mathbf{v} : B$ . Then  $\rho, \xi \models \mathbf{u} : A \rightarrow B$ , i.e.  $\llbracket \mathbf{u} \rrbracket_\rho \in \llbracket A \rightarrow B \rrbracket_\xi = \llbracket A \rrbracket_\xi \rightarrow \llbracket B \rrbracket_\xi$  and  $\llbracket \mathbf{v} \rrbracket_\rho \in \llbracket A \rrbracket_\xi$ . Then  $\llbracket \mathbf{u} \mathbf{v} \rrbracket_\rho = \llbracket \mathbf{u} \rrbracket_\rho \llbracket \mathbf{v} \rrbracket_\rho \in \llbracket B \rrbracket_\xi$ , so  $\rho, \xi \models \mathbf{u} \mathbf{v} : B$ .
1.  $\frac{\Gamma \Vdash \mathbf{u} : A \rightarrow B \quad \Gamma \Vdash \mathbf{v} : A}{\Gamma \Vdash \mathbf{u} \mathbf{v} : B} \rightarrow E^\triangleleft$
- Assume  $\rho, \xi \models \Gamma$  in order to show  $\rho, \xi \models \lambda x \mathbf{t} : A \rightarrow B$ . That is, we must show  $(\llbracket \lambda x \mathbf{t} \rrbracket_\rho \mathbf{s}) \in \llbracket B \rrbracket_\xi$  for all  $\mathbf{s} \in \llbracket A \rrbracket_\xi$ . Suppose  $\mathbf{s} \in \llbracket A \rrbracket_\xi$ , then  $\rho(x := \mathbf{s}) \models \Gamma, x : A$  and hence by the induction hypothesis  $\llbracket \mathbf{t} \rrbracket_{\rho(x := \mathbf{s})} \in B$ . Since  $\llbracket \lambda x \mathbf{t} \rrbracket_\rho \mathbf{s} \equiv ((\lambda x \mathbf{t})[\vec{y} := \rho(\vec{y})] \mathbf{s}) \rightarrow_\beta \mathbf{t}[\vec{y} := \rho(\vec{y}), x := \mathbf{s}] \equiv \llbracket \mathbf{t} \rrbracket_{\rho(x := \mathbf{s})}$ , it follows from the saturation of  $\llbracket B \rrbracket_\xi$  that  $(\llbracket \lambda x \mathbf{t} \rrbracket_\rho \mathbf{s}) \in \llbracket B \rrbracket_\xi$ .
2.  $\frac{\Gamma, x : A \Vdash \mathbf{t} : B}{\Gamma \Vdash \lambda x \mathbf{t} : A \rightarrow B} \rightarrow I^\triangleleft$
- Assume  $\rho, \xi \models \Gamma$  in order to show  $\rho, \xi \models \mathbf{t} : A[T/X]$ . by the induction hypothesis  $\llbracket \mathbf{t} \rrbracket_\rho \in \llbracket \forall X.A \rrbracket_\xi = \bigcap_{Y \in SAT} \llbracket A \rrbracket_{\xi(X := Y)}$ , hence  $\llbracket \mathbf{t} \rrbracket_\rho \in \llbracket A \rrbracket_{\xi(X := \llbracket T \rrbracket_\xi)} = \llbracket A[T/X] \rrbracket_\xi$ .
3.  $\frac{\Gamma \Vdash \mathbf{t} : \forall X.A}{\Gamma \Vdash \mathbf{t} : A[T/X]} \forall E^\triangleleft$
- Assume  $\rho, \xi \models \Gamma$  in order to show  $\rho, \xi \models \mathbf{t} : \forall X.A$ . Since  $X \notin FV(\Gamma)$ , one also has  $\forall Y \in SAT$  that  $\rho, \xi(X := Y) \models \Gamma$ , therefore  $\forall Y \in SAT$ ,  $\llbracket \mathbf{t} \rrbracket_\rho \in \llbracket A \rrbracket_{\xi(X := Y)}$ , then by the induction hypothesis  $\llbracket \mathbf{t} \rrbracket_\rho \in \llbracket \forall X.A \rrbracket_\xi$ , i.e.  $\rho, \xi \models \mathbf{t} : \forall X.A$ .
4.  $\frac{\Gamma \Vdash \mathbf{t} : A \quad X \notin FV(\Gamma)}{\Gamma \Vdash \mathbf{t} : \forall X.A} \forall I^\triangleleft$
- Suppose  $\rho, \xi \models \Gamma$  in order to show  $\rho, \xi \models \mathbf{t} + \mathbf{u} : A$ . By the induction hypothesis one has  $\Gamma \models \mathbf{t} : A$  and  $\Gamma \models \mathbf{u} : A$ , so  $\llbracket \mathbf{t} \rrbracket_\rho \in \llbracket A \rrbracket_\xi$  and  $\llbracket \mathbf{u} \rrbracket_\rho \in \llbracket A \rrbracket_\xi$ . Since  $\llbracket \mathbf{t} + \mathbf{u} \rrbracket_\rho = (\mathbf{t} + \mathbf{u})[\vec{x} := \rho(\vec{x})] = \mathbf{t}[\vec{x} := \rho(\vec{x})] + \mathbf{u}[\vec{x} := \rho(\vec{x})] = \llbracket \mathbf{t} \rrbracket_\rho + \llbracket \mathbf{u} \rrbracket_\rho$ , it follows from the saturation of  $\llbracket A \rrbracket_\xi$  that  $\llbracket \mathbf{t} + \mathbf{u} \rrbracket_\rho \in \llbracket A \rrbracket_\xi$ .
5.  $\frac{\Gamma \Vdash \mathbf{t} : A \quad \Gamma \Vdash \mathbf{u} : A}{\Gamma \Vdash \mathbf{t} + \mathbf{u} : A} +I^\triangleleft$
- Suppose  $\rho, \xi \models \Gamma$  in order to show  $\rho, \xi \models \alpha.\mathbf{t} : A$ . By the induction hypothesis one has  $\Gamma \models \mathbf{t} : A$ , then  $\llbracket \mathbf{t} \rrbracket_\rho \in \llbracket A \rrbracket_\xi$ . Since  $\llbracket \alpha.\mathbf{t} \rrbracket_\rho = (\alpha.\mathbf{t})[\vec{x} := \rho(\vec{x})] = \alpha.(\mathbf{t}[\vec{x} := \rho(\vec{x})]) = \alpha.\llbracket \mathbf{t} \rrbracket_\rho$ , it follows from the saturation of  $\llbracket A \rrbracket_\xi$  that  $\llbracket \alpha.\mathbf{t} \rrbracket_\rho \in \llbracket A \rrbracket_\xi$ .
6.  $\frac{\Gamma \Vdash \mathbf{t} : A}{\Gamma \Vdash \alpha.\mathbf{t} : A} \alpha I^\triangleleft$

**Theorem 4 (Strong normalisation for  $\lambda 2^{la}$ ).**  $\Gamma \Vdash \mathbf{t} : T \Rightarrow \mathbf{t}$  is strongly normalising.

PROOF. Suppose  $\Gamma \Vdash \mathbf{t} : T$ . Then by theorem 3,  $\Gamma \models \mathbf{t} : T$ . Define  $\rho_0(x) = x$  for all  $x$  and let  $\xi$  be a valuation in SAT. Then  $\rho_0, \xi \models \Gamma$  (i.e. for all  $(x : A) \in \Gamma$ ,  $\rho_0, \xi \models x : A$  since  $x \in \llbracket A \rrbracket_\xi$  holds because  $\llbracket A \rrbracket_\xi$  is saturated). Therefore  $\rho_0, \xi \models \mathbf{t} : T$ , hence  $\mathbf{t} \equiv \llbracket \mathbf{t} \rrbracket_{\rho_0} \in \llbracket T \rrbracket_\xi \subseteq SN$ .

It is possible to map every type from *scalar* to a type in  $\lambda 2^{la}$  as follows.

**Definition 10.** Let  $(\cdot)^\sharp$  be a map from  $\mathcal{T} \setminus \{\bar{0}\}$  to  $\mathbb{T}(\lambda 2^{la})$  defined as follows.

$$\begin{aligned} (\alpha.X)^\sharp &= X & (\alpha.\forall X.T)^\sharp &= \forall X.T^\sharp & (\alpha.A \rightarrow B)^\sharp &= A^\sharp \rightarrow B^\sharp \\ (A[B/X])^\sharp &= A^\sharp[B^\sharp/X] & \forall T_1 \equiv T_2, \quad T_1^\sharp &= T_2^\sharp \end{aligned}$$

**Notation**  $\Gamma^\sharp = \{(x : T^\sharp) \mid (x : T) \in \Gamma\}$  and  $\bar{0}^\sharp = T$  for whatever type  $T \in \mathbb{T}(\lambda 2^{la})$ .

We prove that if it is possible to give a type to a term in *scalar* then it is possible to give to the term the mapped type in  $\lambda 2^{la}$ .

**Lemma 19 (Correspondence with  $\lambda 2^{la}$ ).**  $\Gamma \vdash \mathbf{t} : T \Rightarrow \Gamma^\sharp \Vdash \mathbf{t} : T^\sharp$ .

PROOF. Let  $S_n = \Gamma \vdash \mathbf{t} : T$ . We proceed by induction over  $n$ .

Basic cases.  $n = 1$ .

1.  $\frac{}{\Gamma, x:U \vdash x:U} ax$   $(\Gamma, x:U)^{\natural} = \Gamma^{\natural}, x:U^{\natural}$ , so by  $ax^{\natural}$ ,  $(\Gamma, x:U)^{\natural} \Vdash x:U^{\natural}$ .
2.  $\frac{}{\Gamma \vdash \mathbf{0}:\bar{0}} ax\bar{0}$  By  $ax_0^{\natural}$ ,  $\Gamma^{\natural} \Vdash \mathbf{0}:T$  for any  $T \in \mathbb{T}(\lambda 2^{la})$ , so take  $\bar{0}^{\natural} = T$ .

Inductive cases. In all cases, if  $A \equiv \bar{0}$  we can take  $A^{\natural} = T$  for any  $T \in \mathbb{T}(\lambda 2^{la})$  and it is still valid by using the type equivalences.

1.  $\frac{\Gamma \vdash \mathbf{u}:\alpha.(U \rightarrow B) \quad \Gamma \vdash \mathbf{v}:\beta.U}{\Gamma \vdash \mathbf{u} \mathbf{v}:(\alpha \times \beta).B} \rightarrow E$  By the induction hypothesis  $\Gamma^{\natural} \Vdash \mathbf{u}:U^{\natural} \rightarrow B^{\natural}$  and  $\Gamma^{\natural} \Vdash \mathbf{v}:U^{\natural}$ , so by rule  $\rightarrow E^{\natural}$ ,  $\Gamma^{\natural} \Vdash \mathbf{u} \mathbf{v}:B^{\natural} = ((\alpha \times \beta).B)^{\natural}$ .
2.  $\frac{\Gamma, x:U \vdash \mathbf{t}:A}{\Gamma \vdash \lambda x \mathbf{t}:U \rightarrow A} \rightarrow I$  By the induction hypothesis  $\Gamma^{\natural}, x:U^{\natural} \Vdash \mathbf{t}:A^{\natural}$ , so by rule  $\rightarrow I^{\natural}$ ,  $\Gamma^{\natural} \Vdash \lambda x \mathbf{t}:U^{\natural} \rightarrow A^{\natural} = (U \rightarrow A)^{\natural}$ .
3.  $\frac{\Gamma \vdash \mathbf{t}:\forall X.B}{\Gamma \vdash \mathbf{t}:B[U/X]} \forall E$  By the induction hypothesis  $\Gamma^{\natural} \Vdash \mathbf{t}:(\forall X.B)^{\natural} = \forall X.B^{\natural}$ , so by rule  $\forall E^{\natural}$ ,  $\Gamma^{\natural} \Vdash \mathbf{t}:B^{\natural}[U^{\natural}/X]$ .
4.  $\frac{\Gamma \vdash \mathbf{t}:T}{\Gamma \vdash \mathbf{t}:\forall X.T} \forall I$  By the induction hypothesis  $\Gamma^{\natural} \Vdash \mathbf{t}:T^{\natural}$ , so by rule  $\forall I^{\natural}$ ,  $\Gamma^{\natural} \Vdash \mathbf{t}:\forall X.T^{\natural} = (\forall X.T)^{\natural}$ .
5.  $\frac{\Gamma \vdash \mathbf{u}:\alpha.A \quad \Gamma \vdash \mathbf{v}:\beta.A}{\Gamma \vdash \mathbf{u} + \mathbf{v}:(\alpha + \beta).A} +I$  By the induction hypothesis  $\Gamma^{\natural} \Vdash \mathbf{u}:A^{\natural}$  and  $\Gamma^{\natural} \Vdash \mathbf{v}:A^{\natural}$ , so by rule  $+I^{\natural}$ ,  $\Gamma^{\natural} \Vdash \mathbf{u} + \mathbf{v}:A^{\natural} = ((\alpha + \beta).A)^{\natural}$ .
6.  $\frac{\Gamma \vdash \mathbf{t}:A}{\Gamma \vdash \alpha.\mathbf{t}:\alpha.A} sI$  By the induction hypothesis  $\Gamma^{\natural} \Vdash \mathbf{t}:A^{\natural}$ , so by rule  $\alpha I^{\natural}$ ,  $\Gamma^{\natural} \Vdash \alpha.\mathbf{t}:A^{\natural} = (\alpha.A)^{\natural}$ .

Strong normalisation arise as a consequence of strong normalisation for  $\lambda 2^{la}$  and the above lemma.

**Theorem 5 (Strong normalisation).**  $\Gamma \vdash \mathbf{t}:T \Rightarrow \mathbf{t}$  is strongly normalising.

PROOF. By lemma 19,  $\Gamma^{\natural} \Vdash \mathbf{t}:T^{\natural}$ , then by theorem 4,  $\mathbf{t}$  is strong normalising.

Theorem 5 ensures that all the typable terms have a normal form. Taking up again the previous example, terms like  $\mathbf{Y}$  are simply not allowed in this typed setting, as all the terms are strong normalising. So we do not have infinities, and hence the intuitive reasons for having restrictions one on the factorising reduction rules of the Linear-algebraic calculus (*cf.* (\*) in Subsection 2) have now vanished. If we drop them, the example just becomes:

EXAMPLE. Consider some arbitrary typable, and hence normalising term  $\mathbf{t}$ . Then  $\alpha.\mathbf{t} - \alpha.\mathbf{t}$  can be reduced by a factorisation rule into  $(\alpha - \alpha).\mathbf{t}$ . This reduces in one step to  $\mathbf{0}$ , without the need to reduce  $\mathbf{t}$ .

It turns out that in general for typable terms we can indeed drop the restrictions (\*) and (\*\*) that were placed on the factorisation rules and application rules of the operational semantics of *Lineal*, without breaking the confluence of *Lineal*. These restrictions were there only due to the impossibility of checking for the normalisation property in the untyped setting. The full proof of this fact is quite lengthy and at the same time relatively straightforward, as it mainly consists in replacing everywhere in the original proof of the confluence of *Lineal* [3] the *closed normal* assumptions upon terms which arises from using (\*) and (\*\*), by a global normalisation assumption – and check that this works. This fact also reinforces the idea of

that there is a formal correspondence between normalisation in rewriting and expressions of finite norm in algebra.

Having dropped restrictions (\*) and (\*\*) is an important simplification of the linear-algebraic  $\lambda$ -calculus, which becomes really just an oriented version of the axioms of vector spaces [2] together with a linear extension the  $\beta$ -reduction (*i.e.* restriction (\*\*\*) remains of course, to that all function remain linear in their arguments, in the sense of linear-algebra).

## 6. Further properties

### 6.1. A type system for probabilistic calculi

By restricting our scalars to positive reals, the *scalar* type system can be used in order to specialize *Lineal* into a probabilistic calculus. For instance, let us consider the following type judgement, which can be obtained from *scalar*:

$$f ::= \lambda x \left( x \left( \frac{1}{2} \cdot (\text{true} + \text{false}) \right) \left( \frac{1}{4} \cdot \text{true} + \frac{3}{4} \cdot \text{false} \right) \right) : \mathcal{B} \rightarrow \mathcal{B};$$

where  $\mathcal{B}$  stands for  $\forall X. X \rightarrow X \rightarrow X$ . Notice that  $\mathcal{B}$  has true, false, and linear combinations of them with scalars summing to one, as members. Hence in this example the type system provides a guarantee that the function conserves probabilities as summing to one. Indeed, the term can be seen as a probabilistic function such that, if it receives true, it returns a balanced distribution of true and false, but if it receives false, it returns false more frequently than it returns true. We can ask what would the result be if it receives  $\frac{1}{2} \cdot (\text{true} + \text{false})$  and find that everything works as expected, with probabilities summing to one:

$$f \left( \frac{1}{2} \cdot (\text{true} + \text{false}) \right) \longrightarrow^* \frac{3}{8} \cdot \text{true} + \frac{5}{8} \cdot \text{false}.$$

To make this intuition more formal, let us define a type system with the rules and grammar of *scalar*, where the valid types are the classic ones (*i.e.* types exempt of any scalar) and all other types are intermediate types:

**Definition 11.** We define the type system  $\mathcal{P}$  for the probabilistic calculus to be the *scalar* type system with the following restrictions:

- $\mathcal{S} = \mathbb{R}^+$ ,
- Contexts in the type system  $\mathcal{P}$  are sets of tuples  $(x:\mathcal{C})$  such that  $\mathcal{C}$  is in the set  $\mathcal{C} \subsetneq \mathcal{U} \subsetneq \mathcal{T}$  of classical types, that is types exempt of any scalar, which we have also referred to as  $\mathcal{T}^\natural$  in Section 5.
- Type variables run over classical types instead of unit types, *i.e.* the family of  $\forall E[X := C]$  rules accepts only  $C \in \mathcal{C}$ ,
- The final sequent is well-formed in the following sense:  $\forall C \in \mathcal{C}$ , any derivable sequent  $\Gamma \vdash \mathbf{t} : C$  is well-formed, even if the derivation has scalars appearing at intermediate stages.

We define a weight function to check when a term is a probability distribution of terms:

**Definition 12.** Let  $\omega : \Lambda \rightarrow \mathbb{R}^+$  be a function defined inductively by:

$$\begin{aligned} \omega(\mathbf{0}) &= 0 & \omega(\mathbf{t}_1 + \mathbf{t}_2) &= \omega(\mathbf{t}_1) + \omega(\mathbf{t}_2) \\ \omega(\mathbf{b}) &= 1 & \omega(\alpha \cdot \mathbf{t}) &= \alpha \times \omega(\mathbf{t}) \\ \omega(\mathbf{t}_1 \ \mathbf{t}_2) &= \omega(\mathbf{t}_1) \times \omega(\mathbf{t}_2) \end{aligned}$$

where  $\mathbf{b}$  is a base term.

So, we can enunciate the following theorem that shows that every term with a well-formed typing in the type system  $\mathcal{P}$  reduces to a term with weight 1:

**Theorem 6 (Terms in  $\mathcal{P}$  have weight 1).** *Let  $\Gamma \vdash \mathbf{t} : C$  be well-formed, then  $\omega(\mathbf{t} \downarrow) = 1$ .*

PROOF. Instead, we will prove the most general case:  $\Gamma \vdash \mathbf{t} : \alpha.C \Rightarrow \omega(\mathbf{t} \downarrow) = \alpha$ , by structural induction over  $\mathbf{t} \downarrow$ . We take  $\Gamma \vdash \mathbf{t} \downarrow : \alpha.C$ , which is true by theorem 1.

We will need three intermediate results (see appendix Appendix N for their proofs):

**R1:** If  $(\mathbf{t}_1 \ \mathbf{t}_2)$  is in normal form, then  $\mathbf{t}_1 = x$  or  $\mathbf{t}_1 = x \ \vec{\mathbf{r}}$ , where  $\vec{\mathbf{r}} = r_1 \ r_2 \ \dots \ r_n$ .

**R2:**  $\Gamma \vdash x : T \Rightarrow T \in \mathcal{C}$

**R3:**  $\Gamma \vdash x \ \vec{\mathbf{r}} : T$  and  $x \ \vec{\mathbf{r}}$  is in normal form, then  $\exists C \in \mathcal{C}, \alpha \in \mathcal{S}$  such that  $T \equiv \alpha.C$ .

*Basic cases.*

1.  $\mathbf{t} \downarrow = \mathbf{0}$ . Then  $\omega(\mathbf{t} \downarrow) = 0$ . In addition, by lemma 13,  $\alpha = 0$ .
2.  $\mathbf{t} \downarrow = x$ . Then  $\omega(\mathbf{t} \downarrow) = 1$ . In addition, by lemma 12,  $\alpha = 1$ .
3.  $\mathbf{t} \downarrow = \lambda x \ \mathbf{t}'$ . Analogous to 2.

*Inductive cases.*

1.  $\mathbf{t} \downarrow = \gamma.\mathbf{t}'$ . Then  $\omega(\mathbf{t} \downarrow) = \gamma.\omega(\mathbf{t}')$ . By lemma 11,  $\exists U \in \mathcal{U}, \delta \in \mathcal{S}$  such that  $\alpha.C \equiv \gamma.\delta.U$ , and by lemma 2,  $\alpha = \gamma \times \delta$  and there are two options:
  - $\alpha = 0$ , so there are two options:
    - $\gamma = 0$ , then  $\omega(\gamma.\mathbf{t}') = 0 \times \omega(\mathbf{t}') = 0$ , or
    - $\gamma \neq 0, \delta = 0$ , then by lemma 8,  $\Gamma \vdash \mathbf{t}' : 0.U \equiv 0.C$ , so by the induction hypothesis  $\omega(\mathbf{t}') = 0$ , so  $\omega(\gamma.\mathbf{t}') = \gamma \times 0 = 0$ .
  - $\alpha \neq 0$ , then  $C \equiv U$ , so by lemma 8,  $\Gamma \vdash \mathbf{t}' : \delta.C$ . Then by the induction hypothesis  $\omega(\mathbf{t}') = \delta$ . Notice that  $\omega(\mathbf{t} \downarrow) = \gamma \times \omega(\mathbf{t}') = \gamma \times \delta = \alpha$ .
2.  $\mathbf{t} \downarrow = \mathbf{t}_1 + \mathbf{t}_2$ . Then  $\omega(\mathbf{t} \downarrow) = \omega(\mathbf{t}_1) + \omega(\mathbf{t}_2)$ . By lemma 9,  $\exists \sigma, \phi \in \mathcal{S}$  such that

$$\begin{cases} \Gamma \vdash \mathbf{t}_1 : \sigma.C \\ \Gamma \vdash \mathbf{t}_2 : \phi.C \\ \sigma + \phi = \alpha \end{cases}$$

Then by the induction hypothesis  $\omega(\mathbf{t}_1) = \sigma$  and  $\omega(\mathbf{t}_2) = \phi$ , so  $\omega(\mathbf{t}_1) + \omega(\mathbf{t}_2) = \alpha$ .

3.  $\mathbf{t} \downarrow = (\mathbf{t}_1 \ \mathbf{t}_2)$ . Then  $\omega(\mathbf{t} \downarrow) = \omega(\mathbf{t}_1) \times \omega(\mathbf{t}_2)$ . By lemma 6,  $\exists U \in \mathcal{U}, \beta, \gamma, \delta \in \mathcal{S}$  such that

$$\begin{cases} \Gamma \vdash \mathbf{t}_1 : \beta.U \rightarrow \gamma.C \\ \Gamma \vdash \mathbf{t}_2 : \delta.U \\ \beta \times \gamma \times \delta = \alpha \end{cases}$$

As  $(\mathbf{t}_1 \ \mathbf{t}_2)$  is in normal form, by the result **R1**,  $\mathbf{t}_1$  is a variable or a variable applied to something else, so by **R2** and **R3**,  $U \rightarrow \gamma.C \in \mathcal{C}$ , so  $\gamma = 1$  and  $U \in \mathcal{C}$ , then by the induction hypothesis,  $\omega(\mathbf{t}_1) = \beta$  and  $\omega(\mathbf{t}_2) = \delta$ , so  $\omega(\mathbf{t} \downarrow) = \omega(\mathbf{t}_1) \times \omega(\mathbf{t}_2) = \beta \times \delta = \beta \times \gamma \times \delta = \alpha$ .

Notice that, by [3, Proposition 2], closed normal terms have form  $\sum_{i=1}^n \alpha_i.\lambda x \ \mathbf{t}_i + \sum_{j=1}^m \lambda x \ \mathbf{u}_j$ . The above theorem

entails that  $\sum_{i=1}^n \alpha_i + m = 1$ .

Hence the type system  $\mathcal{P}$ , an easy variation of the *scalar* type system, specializes *Lineal* into a probabilistic higher-order  $\lambda$ -calculus.

*Remark.* It is easy to prove that

$$z : U, w : U \vdash ((2.\lambda x \lambda y \frac{1}{4}.x + \frac{1}{4}.y) z) w : U.$$

But notice that  $\omega(((2.\lambda x \lambda y \frac{1}{4}.x + \frac{1}{4}.y) z) w) = 2$ , even when  $((2.\lambda x \lambda y \frac{1}{4}.x + \frac{1}{4}.y) z) w \rightarrow^* \frac{1}{2}.z + \frac{1}{2}.w$ . So, *a priori* this  $\omega$  function cannot tell us that this term will yield a probability distribution of terms (notice that  $\omega$  of the reduced term is 1). However the fact that has type  $U$  in  $\mathcal{C}$ , according to the previous theorem, anticipates this result.

## 6.2. A no-cloning theorem in the logic induced by the type system

A type system always gives rise to a logic: the logical propositions are the types; the sequents are the contexts plus the types; the logical rules are obtained simply by erasing the terms from the typing rules; the proofs are obtained simply by erasing the terms from the type derivation trees – or equivalently by applying the logical rules upon the logical propositions. We call *scalar logic*, and denote  $\mathcal{SL}$  the logic obtained from *scalar*, as defined in Section 5. The present Section shows that proofs in  $\mathcal{SL}$  enjoy a no-cloning property. The potential significance will be discussed in Section 7, although the aware reader will recognize worries related to non-duplication in *Linear Logic* and no-cloning in quantum computation. Informally, this property states that  $\mathcal{SL}$  has no fixed proof method for duplicating a proposition.

First we need to define what we mean by proof method, and for this we need the following lemma.

**Lemma 20 (The rules of  $\mathcal{SL}$  are deterministic).** *Let  $R$  be a  $\mathcal{SL}$  rule and let  $Q_i, Q'_i$ , with  $i = 1, \dots, n$ , be sequents. Then*

$$\left\{ \frac{Q_1, \dots, Q_n}{S} R \wedge \frac{Q'_1, \dots, Q'_n}{S'} R \wedge \forall i, Q_i \equiv Q'_i \right\} \Rightarrow S \equiv S'$$

Hence if  $\Pi$  is a tree with nodes labelled by names of  $\mathcal{SL}$  logical rules, then one may think of  $\Pi$  as a function from sequents to proofs, *i.e.* a proof method:

**Definition 13.** We define recursively the concept of *proof method* of order  $n$  to be the set of functions  $\Pi_n$  which take the following form:

$$\Pi_0(S) = S \\ \Pi_n(S) = \frac{\Pi_{n-1}(S)}{P} R \quad \text{or} \quad \frac{\Pi_k(S) \quad \pi_h}{P} R \quad \text{or} \quad \frac{\pi_k \quad \Pi_h(S)}{P} R$$

where

- $S$  is a sequent,
- $\pi_n$  is a constant proof of size  $n$ ,
- $\max\{k, h\} = n - 1$ ,
- $R$  is a logical rule, and
- $P$  is a sequent such that the resulting proof is *well-formed*.

**Notation** We denote by  $C(\Pi_n(S))$  to the conclusion (root) of the proof  $\Pi_n(S)$ .

A no-cloning theorem can be defined in terms of proof methods, and the way they treat scalars, *i.e.* there is not a generic proof method that is able to take a sequent with a scalar in its type as argument, and then return a sequent where such a scalar appears more than once in the type.

**Theorem 7 (No-cloning of scalars).**  $\nexists \Pi_n$  such that  $\forall \alpha, C(\Pi_n(\Gamma \vdash \alpha.U)) = \Delta \vdash (\delta \times \alpha^s + \gamma).V$  with  $\delta \neq 0$  and  $\gamma$  constants in  $\mathcal{S}$ ,  $s \in \mathbb{N}^{>1}$  and  $U, V$  constants in  $\mathcal{U}$ .

Notice that  $\alpha$  is a member of a ring and  $s$  is a natural number, so  $\alpha^s$  is just the multiplication of  $\alpha$  by itself  $s$  times.

**PROOF.** Induction over  $n$ .

*Basic case.*  $n = 0$ . Trivial, as  $\Pi_0(\Gamma \vdash \alpha.U) = \Gamma \vdash \alpha.U$  for all  $\Pi$ .

*Inductive cases.*

$$\bullet \Pi_n(\Gamma \vdash \alpha.U) = \frac{\Pi_{n-1}(\Gamma \vdash \alpha.U)}{P} R$$

Assume  $P = \Delta \vdash (\delta \times \alpha^s + \gamma).V$  and let us do an analysis case by case on the possible rules  $R$ :

1.  $R = \Rightarrow I[W]$ . Because the denominator must be unit,  $\forall \alpha, \delta \times \alpha^s + \gamma = 1$ , which is a contradiction.
2.  $R = \forall E[X := W]$ . Then  $(\delta \times \alpha^s + \gamma).V = T[X/W]$ , and  $C(\Pi_{n-1}(\Gamma \vdash \alpha.U)) = \forall X.T$ . By lemma 1,  $\exists Z \in \mathcal{U}, \beta \in \mathcal{S}$  such that  $T \equiv \beta.Z$ , so by lemma 2,  $\delta \times \alpha^s + \gamma = \beta$ , then  $C(\Pi_{n-1}(\Gamma \vdash \alpha.U)) = \forall X.\beta.Z = \forall X.(\delta \times \alpha^s + \gamma).Z \equiv (\delta \times \alpha^s + \gamma).\forall X.Z$ , which is a contradiction by the induction hypothesis.
3.  $R = \forall I[X]$ . Then  $(\delta \times \alpha^s + \gamma).V \equiv \forall X.T$ . Analogous to 2.
4.  $R = sI[\beta]$ . Then  $\delta \times \alpha^s + \gamma.V \equiv \beta.T$ . By lemma 1,  $T \equiv \sigma.W$ , then by lemma 2,  $\delta \times \alpha^s + \gamma = \beta \times \sigma$ . Notice that  $\beta$  cannot depend on  $\alpha$  as the rule is constant, so it must be  $\sigma$  depending on  $\alpha^s$ , which is a contradiction by the induction hypothesis.

$$\bullet \Pi_n(\Gamma \vdash \alpha.U) = \frac{\Pi_k(\Gamma \vdash \alpha.U) \quad \pi_h}{P} R$$

Assume  $P = \Delta \vdash (\delta \times \alpha^s + \gamma).V$  and let us do an analysis case by case on the possible rules  $R$ :

1.  $R = \Rightarrow E$ . Then  $C(\pi_h) = \Delta \vdash \beta.W$  and  $C(\Pi_k(\Gamma \vdash \alpha.U)) = \Delta \vdash \phi.W \rightarrow \sigma.V$  where  $\forall \alpha, \beta \times \phi \times \sigma = \delta \times \alpha^s + \gamma$ .  $\beta$  cannot depend on  $\alpha$ , as  $\pi_h$  is constant, so:
  - Assume  $\phi$  depend on  $\alpha$ , and  $\sigma$  do not, then it depend linearly on  $\alpha$  by the induction hypothesis.
  - Assume  $\sigma$  depend on  $\alpha$ , then there are two possibilities:
    - (a)  $U$  is an arrow with the last term of the arrow being  $\sigma.V$ , which is a contradiction as  $\sigma$  depend on  $\alpha$  and  $U$  is fixed.
    - (b) The arrow is set up through the derivation, so at some point we must had to use  $\rightarrow I$  rule in the following way

$$\frac{\Theta, Z \vdash \sigma.V}{\Theta \vdash Z \rightarrow \sigma.V} \rightarrow I$$

so by the induction hypothesis  $\sigma$  depends linearly on  $\alpha$ . Once we reach this point, the only possibility to add something depending on  $\alpha$  and multiplying the whole type is with  $sI[\alpha]$  as it cannot come from any other branch (all other branches are constants). However, it is not possible either, as all the rules must to be constants.

2.  $R = +I$ . Then  $C(\Pi_k(\Gamma \vdash \alpha.U)) = \Delta \vdash \sigma.V$  and  $C(\pi_h) = \Delta \vdash \phi.V$  where  $\sigma + \phi = \delta \times \alpha^2 + \gamma$ . So, as  $\phi$  is constant,  $\sigma = \delta \times \alpha^2 + \gamma - \phi$ , which is a contradiction by the induction hypothesis.

$$\bullet \Pi_n(\Gamma \vdash \alpha.U) = \frac{\pi_k \quad \Pi_h(\Gamma \vdash \alpha.U)}{P} R$$

Assume  $P = \Delta \vdash (\delta \times \alpha^s + \gamma).V$  and let us do an analysis case by case on the possible rules  $R$ :

1.  $R = \Rightarrow E$ . Then  $C(\pi_k) = \Delta \vdash \phi.W \rightarrow \sigma.V$  and  $C(\Pi_h(\Gamma \vdash \alpha.U)) = \beta.W$  where  $\beta \times \phi \times \sigma = \delta \times \alpha^s + \gamma$ . Notice that nor  $\phi$  nor  $\sigma$  can depend on  $\alpha$ , so the only possibility is to  $\beta$  to depend on  $\alpha^s$ , which is a contradiction by the induction hypothesis.
2.  $R = +I$ . Analogous 2 of the previous case.

We can reformulate this theorem to look more like a no-cloning theorem in the following way<sup>2</sup>.

---

<sup>2</sup>Where  $T \otimes T$  stands for the usual encoding of tuples. Formally, to allow such an encoding for general types, we need to add the following equivalence  $(\alpha.U) \rightarrow T \equiv \alpha.(U \rightarrow T)$ , as was discussed in Section 3



**Corollary 3 (No-cloning Theorem).**  $\nexists \Pi_n$  such that  $\forall T \in \mathcal{T}, \Pi_n(\Gamma \vdash T)$  is a witness of  $\Gamma \vdash T \Rightarrow \Delta \vdash T \otimes T$ .

PROOF. By lemma 1,  $\exists \alpha \in \mathcal{S}, U \in \mathcal{U}$  such that  $T \equiv \alpha.U$ , so  $T \otimes T \equiv \alpha.U \otimes \alpha.U \equiv \alpha^2.(U \otimes U) = (1 \times (1 \times \alpha^2 + 0)).(U \otimes U)$ . Then by theorem 7 the corollary holds.

Hence our no-cloning allows the existence of a proof method  $\Pi$  such that  $\Pi(\Gamma \vdash T)$  has conclusion  $\Gamma \vdash T \Rightarrow \Delta \vdash T \otimes T$ , but it does not allow the *same* proof method  $\Pi$  to accomplish this for any proposition  $T$ .

## 7. Discussion, future works, prospects

### 7.1. Expressiveness of probabilistic calculi

In Subsection 6.1 we have shown how an easy variant of the *scalar* type system specializes *Lineal* into a higher-order probabilistic calculus, but we have hardly studied this probabilistic calculus. For instance we have proved that it expresses probabilistic functions, but have not identified which class of probabilistic functions. Some ongoing work already suggests that there are ways of widening the class of probabilistic functions that can be expressed by extending the *scalar* type system with a ‘sum of types’ construct (e.g.  $2.A + B$ ). Hence we postpone this important discussion of expressiveness till future work on a *vectorial* type system.

### 7.2. Relation with Linear Logic, no-cloning and the quantum

In Subsection 6.2 we have defined  $\mathcal{SL}$ , the logic induced by the *scalar* type system when we forget about terms. The propositions in this logic are weighted by scalars (e.g.  $U \rightarrow (2/3).V$ ), but what is the meaning that one can attach to these scalars? The  $+I$  rule suggests that we need two proofs of  $A$  present in the proof tree in order to prove  $2.A$ . Hence scalars in  $\mathcal{SL}$  seem to reflect the quantity of proofs of atomic propositions that are needed to prove the composite proposition. However, this ‘proof counting’ interpretation holds true only after two modifications. First, we need remove the family of rules  $sI[\cdot]$ , because it trivially allows us to prove  $2.A$  from one proof of  $A$ . But this jeopardizes subject reduction (specifically in the rule  $\alpha.\mathbf{u} + \beta.\mathbf{u} \rightarrow (\alpha + \beta).\mathbf{u}$ ), so we need to add an alternative typing rule such as

$$\frac{\Gamma \vdash \alpha.\mathbf{u}:\delta.T \quad \Gamma \vdash \beta.\mathbf{u}:\gamma.T}{\Gamma \vdash (\alpha + \beta).\mathbf{u}:(\delta + \gamma).T}$$

From this alternative rule we are able to derive  $sI[\cdot]$  for integer scalars as a theorem, but in this process we would have to repeat the proofs of  $A$   $\alpha$  times in order to get to  $\alpha.A$ . Secondly, let  $A \equiv \alpha.U$  and consider the *duplicator*  $\vdash \lambda x 2.x:\forall X.X \rightarrow 2.X$ , which allows:

$$\frac{\frac{\frac{}{\vdash \lambda x 2.x:\forall X.X \rightarrow 2.X}}{\vdash \lambda x 2.x:U \rightarrow 2.U} \quad \forall E[X := U] \quad \frac{\vdots}{\vdash \mathbf{t}:\alpha.U \equiv A}}{\vdash (\lambda x 2.x) \mathbf{t}:2.\alpha.U \equiv 2.A} \rightarrow E$$

without needing to prove  $A$  twice. Hence the proof-counting interpretation ought to hold, but only after cut-elimination; *i.e.* the removal of all  $\rightarrow E$  in the derivation tree.

This idea of counting proofs, and hence considering them as resources, is reminiscent of *bounded linear logic* ( $\mathcal{BLL}$ ) [20], and more generally *linear logic* ( $\mathcal{LL}$ ) [19]. However these do not only *count* the amount of resources available, they also make it impossible to add new resources. In  $\mathcal{LL}$  the context puts a definite limit on how many resources we can use, whereas this is not the case in  $\mathcal{SL}$ . Since the  $\mathcal{SL}$  ‘counts proofs’ whereas  $\mathcal{LL}$  ‘counts and limits proofs’, this suggests that  $\mathcal{SL}$  may be embedded in  $\mathcal{LL}$ . Ongoing works confirm this intuition: when scalars are restricted to be integer numbers,  $\mathcal{SL}$  can indeed be encoded as a

fragment of IMELL, a subset of  $\mathcal{LL}$  [12]. The encoding is likely to only be an abstract interpretation when scalars are not restricted to integers.

Clearly  $\mathcal{SL}$ , unlike  $\mathcal{LL}$ , does not refrain us from duplicating resources. Yet in Subsection 6.2 we have been able to prove a no-cloning theorem for  $\mathcal{SL}$ . How can we make sense of this apparent contradiction? Consider the *copying machine*  $\vdash \lambda x x \otimes x : \forall X. X \rightarrow X \otimes X$ , and let  $A \equiv \alpha.U$ , then this machine allows:

$$\frac{\frac{\frac{\vdash \alpha. \lambda x x \otimes x : \alpha. \forall X. X \rightarrow (X \otimes X)}{\vdash \alpha. \lambda x x \otimes x : \alpha. U \rightarrow U \otimes U} \quad \forall E[X := U] \quad \frac{\vdash \mathbf{t} : \alpha. U \equiv A}{\vdash \mathbf{t} : \alpha. U \equiv A}}{\vdash (\alpha. \lambda x x \otimes x) \mathbf{t} : \alpha^2. (U \otimes U) \equiv A \otimes A} \rightarrow E$$

This proof tree that yields  $A \otimes A$  from a single proof of  $A$ , which needs be plugged as the right branch of the tree. However the symbol  $A$  appears also in the right branch of the tree; hence the proof method that duplicates  $A$  crucially depends on  $A$ . It is on this basis that our no-cloning theorem is formulated; our no-cloning allows the existence of a proof method  $\Pi$  such that  $\Pi(\Gamma \vdash T)$  has conclusion  $\Gamma \vdash T \Rightarrow \Delta \vdash T \otimes T$ , but it does not allow the *same* proof method  $\Pi$  to work for any type. This way of phrasing no-cloning must probably hold in  $\mathcal{LL}$  as well, but it is not usually contemplated.  $\mathcal{SL}$  emphasizes this property, which we believe is much more in line with quantum theory than the straightforward non-duplication of resources of  $\mathcal{LL}$ . Indeed, quantum theory states that it is not possible to have a *universal* cloning machine, but does allow cloning machines of *specific* vectors.

### 7.3. Towards a quantum physical logic from Curry-Howard?

The original motivation behind *Lineal* was to seek to capture the underlying structures behind Quantum Computation. This was achieved to some extent, since any quantum algorithm can be expressed in *Lineal* [3]. But to some extent this has not yet been achieved, because in the untyped calculus one can express non-unitary, and hence non-physical linear operators. The problem of finding a type system that specializes *Lineal* into a quantum programming language is a subject for future works. Nevertheless it is clear that this problem is very much alike the one of checking for preservation of probabilities, and hence the type system given here is certainly a contribution in that direction. Hence from this perspective, our paper can be viewed as part of a larger trend [32, 37, 39, 5, 25, 31, 1, 21] towards developing quantum programming languages [17, 30]. Of course one of the purposes of such quantum programming languages would be to express quantum programs in an elegant manner, but we believe that this is not a good enough reason – as not many quantum algorithms are known. A more important reason in our view is to provide a theoretical framework, *i.e.* a common and formal language, for reasoning and proving properties about these quantum algorithms and quantum information processing applications in general.

Indeed, on the one hand there is this clear need for a logic that could aid us in isolating the reasoning behind some quantum algorithms; *i.e.* that would provide a tool to explore whether or not there is some typically ‘quantum piece of thinking’ behind some algorithms such as Grover’s [22] and Shor’s [33] – which remain somewhat unintuitive. On the other hand it is clear also that classical computer science has now got a long experience of expressing the reasoning behind a program via several formally-defined logics, and that often these logics arise via the study of type systems for the programming language – through what has become known as the Curry-Howard isomorphism [34]. Hence, rather than coming up with some *ad hoc* logics that would only reflect our current lack of understanding of the deep nature of quantum information (which is how many authors feel about ‘quantum logic’ [7]), we would like to get to such a logic progressively and legitimately, from the study of the type system of a quantum programming language. In other words, this long-term program could be summarized as follows: we have programming languages, we know that the Curry-Howard correspondence builds logics from typed programming languages, so what non-trivial logic can it yield if applied to quantum programming languages? At the moment we have scalars within the types, and we have managed to give a statement of the no-cloning theorem solely in terms of types (see section 6), which is promising. In the future we may have sums within types, and hence a vector space

of types hopefully reflecting more of the properties of quantum information. Though there may be other routes for fine-graining our type systems and capturing such properties, as was illustrated for quantifying entanglement by [27, 29].

#### 7.4. The model-oriented approach

In the denotational semantics approach to typed calculi, terms are understood as functions, and types are understood as sets of functions, over a well-known mathematical space. This understanding provides an alternative path to new type system design: one can start by thinking of sets of functions that need to be characterized, and then work out a type system that accomplishes the job. Arguably this path can lead to less syntactic, more meaningful proofs of the properties of the type system, sustained by our guiding intuition of the underlying mathematical space. On the other hand, the complexity of the proofs in this paper is largely due to the large number of rules (16 rules plus associativity and commutativity of  $+$ ), a difficulty which seems hard to circumvent. Moreover the issue of models of (Linear-)Algebraic  $\lambda$ -calculus is a challenging, active topic of current research. We know of the categorical model of simply typed *Lineal* [38], and the finiteness space model of simply typed Algebraic  $\lambda$ -calculus [14, 36]. Whilst revising this paper, a syntactic finite space model System  $F$  algebraic  $\lambda$ -calculus has been developed in [15]. Hence known models are intricate and tend not to cover the set of terms under consideration in this paper. Notice also that since the models of untyped  $\lambda$ -calculus are uncountable vector spaces, the models of (Linear-)Algebraic  $\lambda$ -calculus are likely to be uncountable vector space. These are fascinating, open questions.

## 8. Summary of contributions

In summary, we have defined a System  $F$ -like type system for an extension of *Lineal*, a  $\lambda$ -calculus which allows making arbitrary linear combinations of  $\lambda$ -calculus terms  $\alpha.\mathbf{t} + \beta.\mathbf{u}$ . This *Scalar* type system is fine-gained in that it keeps track of the ‘amount of a type’, *i.e.* the type of terms contain a scalar which is the sum of the amplitudes of the terms which contribute to this type.

Our main technical contributions were:

- A proof of the subject reduction property of this *scalar* type system (theorem 1). This came out after having proven a set of lemmas related to the equivalence relation intrinsic to the types, and another set of lemmas explaining how the scalars within the types are related to the scalars within the terms. Once all of the important properties were known, we were able to use them to decompose and recompose any term before and after applying a reduction rule, so as to show that every reduction rule preserves the types.
- A proof of the strong normalisation property of this *scalar* type system (theorem 5). The technique used to prove the strong normalisation property was by proving that such property would hold for a simpler system, and then to show the correspondence between the two systems. As a direct consequence of this property, some restrictions were lifted in the reduction rules, allowing the factorisation not only of closed normal terms but also of strong normalising terms – which is the case of all the typable terms.

In the discussion we derived two other important results:

- We have explained and demonstrated that the *scalar* type system can readily be made into a type system for probabilistic calculi, which specializes the calculus so that the functions thereby defined are guaranteed to be acceptable probabilistic functions.
- We have begun to look at the logic induced by this *scalar* type system and formulated a no-cloning theorem (corollary 3) solely in terms of proof methods in that logic.

We have discussed the potential impact and follow-up to these contributions in Section 7.

## Acknowledgements

We would like to thank to Gilles Dowek, Jonathan Grattage, Philippe Jorrand, Simon Perdrix, Barbara Petit, Frédéric Prost and Benoît Valiron for enlightening discussions.

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## Appendices

### Appendix A. Proof of lemma 1

Let  $\mu(\cdot): \mathcal{T} \rightarrow \mathbb{N}_0$  be a map defined inductively by

$$\begin{aligned} \mu(X) &= 0 & \mu(\forall X. \mathcal{T}) &= 1 + \mu(\mathcal{T}) \\ \mu(\mathcal{U} \rightarrow \mathcal{T}) &= 0 & \mu(\alpha. \mathcal{T}) &= 1 + \mu(\mathcal{T}) \\ \mu(\overline{0}) &= 0 \end{aligned}$$

Then we proceed by induction over  $\mu(T)$ .

*Basic cases.* Let  $\mu(T) = 0$ . Then

1.  $T = X$ , then  $T \in \mathcal{U}$  and  $T \equiv 1.T$ .
2.  $T = \mathcal{U} \rightarrow A$ , then  $T \in \mathcal{U}$  and  $T \equiv 1.T$ .
3.  $T = \overline{0}$ , then  $\forall U \in \mathcal{U}, T \equiv 0.U$ .

*Inductive cases.* Let  $\mu(T) = n$  and assume the lemma is valid for all  $A$  with  $\mu(A) < n$ . Then, the possible cases are

1.  $T = \forall X. A$ , then  $\mu(A) = n - 1$ , and so by the induction hypothesis  $\exists U \in \mathcal{U} \text{ s.t. } A \equiv U \text{ or } A \equiv \alpha.U$ , then  $T \equiv \forall X. U \in \mathcal{U} \text{ or } T \equiv \forall X. \alpha.U \equiv \alpha. \forall X. U$ .
2.  $T = \alpha. A$ , then  $\mu(A) = n - 1$ , and so by the induction hypothesis  $\exists U \in \mathcal{U} \text{ s.t. } A \equiv U \text{ or } A \equiv \beta.U$ , then  $T \equiv \alpha.U \text{ or } T \equiv \alpha. \beta.U \equiv (\alpha \times \beta).U$ .

### Appendix B. Proof of lemma 2

Following  $\mathcal{U}$  grammar, neither  $U$  nor  $U'$  could contain scalars in this head form but only in the right side of a type  $U \rightarrow T$ . However, no equivalence rule lets it come out from the right of the arrow and get to the head-form, so if  $\alpha.U \equiv \beta.U'$  that means  $\alpha = \beta = 0$  or  $U \equiv U'$  and  $\alpha = \beta$ .

### Appendix C. Proof of lemma 3

$T \geq T'$ , then assume  $T \equiv R_1 > \dots > R_n \equiv T'$ . Then  $\forall i$  one has  $R_i > R_{i+1}$ . So, the possible cases are:

- $R_{i+1} \equiv \forall X. R_i$ , then  $\alpha. R_i > \forall X. \alpha. R_i \equiv \alpha. \forall X. R_i \equiv \alpha. R_{i+1}$ .
- $R_i \equiv \forall X. C$  and  $R_{i+1} \equiv C[U/X]$ , then  $\alpha. R_i \equiv \alpha. \forall X. C \equiv \forall X. \alpha. C > (\alpha. C)[U/X] \equiv \alpha. (C[U/X]) \equiv \alpha. R_{i+1}$ .

## Appendix D. Proof of lemma 4

A map  $(\cdot)^\circ$  is defined by

$$X^\circ = X \quad (A \rightarrow B)^\circ = A \rightarrow B \quad (\forall X.A)^\circ = A^\circ \quad (\alpha.A)^\circ = \alpha.A^\circ \quad (\bar{0})^\circ = \bar{0}$$

We need two intermediate results.

1. Given  $T, U, \exists V / (T[U/X])^\circ \equiv T^\circ[V/X]$
2.  $T \geq R \Rightarrow \exists \vec{U}, \vec{X} / R^\circ \equiv T^\circ[\vec{U}/\vec{X}]$

Proofs

1. Structural induction on  $T$

$T \equiv X$ , then  $(X[U/X])^\circ = U^\circ = X[U^\circ/X] = X^\circ[U^\circ/X]$ .

$T \equiv Y$ , then  $(Y[U/X])^\circ = Y = Y^\circ[U/X]$ .

$T \equiv A \rightarrow B$ , then  $((A \rightarrow B)[U/X])^\circ = (A[U/X] \rightarrow B[U/X])^\circ = A[U/X] \rightarrow B[U/X] = (A \rightarrow B)[U/X] = (A \rightarrow B)^\circ[U/X]$ .

$T \equiv \forall Y.T'$ , then  $((\forall Y.T')[U/X])^\circ = (\forall Y.T'[U/X])^\circ = (T'[U/X])^\circ$ , which is, by the induction hypothesis, equivalent to  $T'^\circ[V/X] = (\forall Y.T')^\circ[V/X]$ .

$T \equiv \bar{0}$ , analogous to  $T \equiv Y$ .

$T \equiv \alpha.T'$ , then  $(\alpha.T'[U/X])^\circ = \alpha.(T'[U/X])^\circ$ , which is, by the induction hypothesis, equivalent to  $\alpha.(T')^\circ[V/X] = ((\alpha.T')^\circ)[V/X]$ .

2. It suffices to show this for  $T > R$ .

Case 1.  $R \equiv \forall X.T$ . Then  $R^\circ \equiv T^\circ$ .

Case 2.  $T \equiv \forall X.S$  and  $R \equiv S[U/X]$  then by the intermediate result 1 one has  $R^\circ \equiv S^\circ[U/X] \equiv T^\circ[U/X]$ .

Proof of the lemma.  $U \rightarrow T \equiv (U \rightarrow T)^\circ$ , by the intermediate result 2, one has this equivalent to  $(V \rightarrow R)^\circ[U/X] \equiv (V \rightarrow R)[U/X]$ .

## Appendix E. Proof of lemma 10. Item 1

Let  $S_n = \Gamma \vdash \mathbf{u} : T$ . Induction over  $n$ .

*Basic cases.*  $n = 0$ .

1.  $\frac{}{\Gamma, x : V \vdash x : V} ax$  Notice that  $(\Gamma, x : V)[U/X] = \Gamma[U/X], x : V[U/X]$ , then by  $ax$  rule,  $(\Gamma, x : V)[U/X] \vdash x : V[U/X]$ .
2.  $\frac{}{\Gamma \vdash \mathbf{0} : \bar{0}} ax_{\bar{0}}$  Notice that  $\bar{0} = \bar{0}[U/X]$ , then by  $ax_{\bar{0}}$ ,  $\Gamma[U/X] \vdash \mathbf{0} : \bar{0}[U/X]$ .

*Inductive cases.*

1.  $\frac{\Gamma \vdash \mathbf{u} : \alpha.(V \rightarrow T) \quad \Gamma \vdash \mathbf{v} : \beta.V}{\Gamma \vdash (\mathbf{u} \ \mathbf{v}) : (\alpha \times \beta).T} \rightarrow E$  by the ind. hypothesis  $\Gamma[U/X] \vdash \mathbf{u} : (\alpha.(V \rightarrow T))[U/X]$ . However, notice that  $(\alpha.(V \rightarrow T))[U/X] \equiv \alpha.V[U/X] \rightarrow T[U/X]$ .  
 $\Gamma[U/X] \vdash \mathbf{u} : (\beta.V)[U/X]$ , however,  $(\beta.V)[U/X] \equiv \beta.V[U/X]$ , so

$$\frac{\Gamma[U/X] \vdash \mathbf{u} : \alpha.V[U/X] \rightarrow T[U/X] \quad \Gamma[U/X] \vdash \mathbf{u} : \beta.V[U/X]}{\Gamma[U/X] \vdash (\mathbf{u} \ \mathbf{v}) : (\alpha \times \beta).T[U/X]} \rightarrow E$$

Notice that  $(\alpha \times \beta).T[U/X] \equiv ((\alpha \times \beta).T)[U/X]$ .

2.  $\frac{\Gamma, x:V \vdash \mathbf{t}:T}{\Gamma \vdash \lambda x \mathbf{t}:V \rightarrow T} \rightarrow I[V]$  by the induction hypothesis  $(\Gamma, x:V)[U/X] \vdash \mathbf{t}:T[U/X]$ . Notice that  $(\Gamma, x:V)[U/X] = \Gamma[U/X], x:V[U/X]$ , then  

$$\frac{\Gamma[U/X], x:V[U/X] \vdash \mathbf{t}:T[U/X]}{\Gamma[U/X] \vdash \lambda x \mathbf{t}:V[U/X] \rightarrow T[U/X]} \rightarrow I[V[U/X]]$$
- Notice that  $V[U/X] \rightarrow T[U/X] \equiv (V \rightarrow T)[U/X]$ .
3.  $\frac{\Gamma \vdash \mathbf{u}:\forall Y.T}{\Gamma \vdash \mathbf{u}:T[V/Y]} \forall E$  by the induction hypothesis  $\Gamma[U/X] \vdash \mathbf{u}:(\forall Y.T)[U/X]$  where  $X \neq Y$  and  $Y \notin FV(U)$ . Then  $(\forall Y.T)[U/X] = \forall Y.T[U/X]$ , and so by using  $\forall E$  rule,  $\Gamma[U/X] \vdash \mathbf{u}:(T[U/X])[W/Y]$ . As  $Y \notin FV(U)$ , then  $(T[U/X])[W/Y] = T[U/X, W/Y]$ . Take  $W = V[U/X]$ , then  $T[U/X, W/Y] = (T[V/Y])[U/X]$ .
4.  $\frac{\Gamma \vdash \mathbf{u}:T}{\Gamma \vdash \mathbf{u}:\forall Y.T} \forall I$  Take  $Z \neq X$  s.t  $Z$  does not appear in  $\Gamma, T, U$ . by the induction hypothesis  $\Gamma[Z/Y] \vdash \mathbf{u}:T[Z/Y]$ , but as  $Y \notin FV(\Gamma)$ , we can just write  $\Gamma \vdash \mathbf{u}:T[Z/Y]$ . by the induction hypothesis again  $\Gamma[U/X] \vdash \mathbf{u}:(T[Z/Y])[U/X]$ . As  $Z$  does not appear in  $\Gamma$  not  $U$ ,  $Z$  does not appear in  $\Gamma[U/X]$ . Then by using  $\forall I$  rule,  $\Gamma[U/X] \vdash \mathbf{u}:\forall Z.((B[Z/Y])[U/X])$ . Notice that  $\forall Z.((B[Z/Y])[U/X]) = (\forall Z.B[Z/Y])[U/X] = (\forall Y.B)[U/X]$ .
5.  $\frac{\Gamma \vdash \mathbf{t}:T}{\Gamma \vdash \alpha.\mathbf{t}:\alpha.T} sI[\alpha]$  by the induction hypothesis  $\Gamma[U/X]:\mathbf{t}:T[U/X]$ , then by using  $sI[\alpha]$  rule,  $\Gamma[U/X]:\alpha.\mathbf{t}:\alpha.T[U/X]$ . Notice that  $\alpha.T[U/X] = (\alpha.T)[U/X]$ .
6.  $\frac{\Gamma \vdash \mathbf{u}:\alpha.A \quad \Gamma \vdash \mathbf{v}:\beta.A}{\Gamma \vdash \mathbf{u}+\mathbf{v}:(\alpha+\beta).A} +I$  by the induction hypothesis  $\Gamma[U/X] \vdash \mathbf{u}:(\alpha.A)[U/X]$  and  $\Gamma[U/X] \vdash \mathbf{v}:(\beta.A)[U/X]$ . And notice that  $(\alpha.A)[U/X] = \alpha.A[U/X]$  and  $(\beta.A)[U/X] = \beta.A[U/X]$ . So, by rule  $+I$ ,  $\Gamma[U/X] \vdash \mathbf{u}+\mathbf{v}:(\alpha+\beta).A[U/X]$ . Notice that  $(\alpha+\beta).A[U/X] = ((\alpha+\beta).A)[U/X]$ .

## Appendix F. Proof of lemma 10. Item 2

Let  $S_n = \Gamma, x:U \vdash \mathbf{t}:T$ . Induction over  $n$ .

*Basic cases.*  $n = 0$ .

1.  $\frac{}{\Gamma, x:U \vdash x:U} ax$  Notice that  $x[\mathbf{b}/x] = \mathbf{b}$ , so  $\Gamma \vdash \mathbf{b}:U$ .
2.  $\frac{}{\Gamma, y:V, x:U \vdash y:V} ax$  Notice that  $y[\mathbf{b}/x] = y$ , so  $\Gamma, y:V \vdash y[\mathbf{b}/x]:V$  by rule  $ax$ .
3.  $\frac{}{\Gamma, x:U \vdash \mathbf{0}:\bar{0}} ax\bar{0}$  Notice that  $\mathbf{0}[\mathbf{b}/x] = \mathbf{0}$ , so  $\Gamma \vdash \mathbf{0}[\mathbf{b}/x]:\bar{0}$  by rule  $ax\bar{0}$ .

*Inductive cases.*

1.  $\frac{\Gamma, x:U \vdash \mathbf{u}:\alpha.(V \rightarrow T) \quad \Gamma, x:U \vdash \mathbf{v}:\beta.V}{\Gamma, x:U \vdash (\mathbf{u} \mathbf{v}):(\alpha \times \beta).T} \rightarrow E$  by the induction hypothesis  $\Gamma \vdash \mathbf{u}[\mathbf{b}/x]:\alpha.(V \rightarrow T)$  and  $\Gamma \vdash \mathbf{v}[\mathbf{b}/x]:\beta.V$ , so using rule  $\rightarrow E$ ,  $\Gamma \vdash (\mathbf{u}[\mathbf{b}/x] \mathbf{v}[\mathbf{b}/x]):(\alpha \times \beta).T$ . Notice that  $(\mathbf{u}[\mathbf{b}/x] \mathbf{v}[\mathbf{b}/x]) = (\mathbf{u} \mathbf{v})[\mathbf{b}/x]$ .
2.  $\frac{\Gamma, x:U \vdash \mathbf{t}:T}{\Gamma \vdash \lambda x \mathbf{t}:U \rightarrow T} \rightarrow I[U]$  Notice that the conclusion on this derivation does not match with the hypothesis, as  $x$  is not in the context. So, the only way to use this lemma ending with a  $\rightarrow I[U]$  rule is as in the following case.

3.  $\frac{\Gamma, y:V, x:U \vdash \mathbf{t}:T}{\Gamma, x:U \vdash \lambda y \mathbf{t}:V \rightarrow T} \rightarrow I[V]$  by the induction hypothesis  $\Gamma, y:V \vdash \mathbf{t}[\mathbf{b}/x]:T$ , then by rule  $\rightarrow I[V]$ ,  $\Gamma \vdash \lambda y (\mathbf{t}[\mathbf{b}/x]):V \rightarrow T$ . Notice that  $\lambda y (\mathbf{t}[\mathbf{b}/x]) = (\lambda y \mathbf{t})[\mathbf{b}/x]$ .
4.  $\frac{\Gamma, x:U \vdash \mathbf{u}:\forall X.T}{\Gamma, x:U \vdash \mathbf{u}:T[V/X]} \forall E$  by the induction hypothesis  $\Gamma \vdash \mathbf{u}[\mathbf{b}/x]:\forall X.T$ , then by rule  $\forall E$ ,  $\Gamma \vdash \mathbf{u}[\mathbf{b}/x]:T[V/X]$ .
5.  $\frac{\Gamma, x:U \vdash \mathbf{u}:T}{\Gamma, x:U \vdash \mathbf{u}:\forall X.T} \forall I$  by the induction hypothesis  $\Gamma \vdash \mathbf{u}[\mathbf{b}/x]:T$ , then by rule  $\forall I$ ,  $\Gamma \vdash \mathbf{u}[\mathbf{b}/x]:\forall X.T$ .
6.  $\frac{\Gamma, x:U \vdash \mathbf{t}:T}{\Gamma, x:U \vdash \alpha.\mathbf{t}:\alpha.T} sI[\alpha]$  by the induction hypothesis  $\Gamma \vdash \mathbf{t}[\mathbf{b}/x]:T$ , then by rule  $sI[\alpha]$ ,  $\Gamma \vdash \alpha.(\mathbf{t}[\mathbf{b}/x]):\alpha.T$ . Notice that  $\alpha.(\mathbf{t}[\mathbf{b}/x]) = (\alpha.\mathbf{t})[\mathbf{b}/x]$ .
7.  $\frac{\Gamma, x:U \vdash \mathbf{u}:\alpha.A \quad \Gamma, x:U \vdash \mathbf{v}:\beta.A}{\Gamma, x:U \vdash \mathbf{u} + \mathbf{v}:(\alpha + \beta).A} +I$  by the induction hypothesis  $\Gamma \vdash \mathbf{u}[\mathbf{b}/x]:\alpha.A$  and  $\Gamma \vdash \mathbf{v}[\mathbf{b}/x]:\beta.A$ , so by rule  $+I$ ,  $\Gamma \vdash \mathbf{u}[\mathbf{b}/x] + \mathbf{v}[\mathbf{b}/x]:(\alpha + \beta).A$ . Notice that  $\mathbf{u}[\mathbf{b}/x] + \mathbf{v}[\mathbf{b}/x] = (\mathbf{u} + \mathbf{v})[\mathbf{b}/x]$ .

## Appendix G. Proof of lemma 12

Let  $S_n = \Gamma \vdash \mathbf{b}:T$ . Induction over  $n$

*Basic case.*  $n = 0$ .

$$\frac{}{\Gamma, x:U \vdash x:U} ax \text{ with } U \in \mathcal{U}$$

*Inductive cases.* The possible cases are

1.  $\frac{\Gamma, x:U \vdash \mathbf{t}:B}{\Gamma \vdash \lambda x \mathbf{t}:U \rightarrow B} \rightarrow I[U]$  with  $U \in \mathcal{U}$  As  $U \in \mathcal{U}$ , then  $U \rightarrow B \in \mathcal{U}$ .
2.  $\frac{\Gamma \vdash \mathbf{b}:\forall X.B}{\Gamma \vdash \mathbf{b}:B[U/X]} \forall E$  by the induction hypothesis  $\forall X.B \in \mathcal{U}$ , so  $B \in \mathcal{U}$  and then  $B[U/X] \in \mathcal{U}$ .
3.  $\frac{\Gamma \vdash \mathbf{t}:B}{\Gamma \vdash \mathbf{t}:\forall X.B} \forall I$  by the induction hypothesis  $B \in \mathcal{U}$ , so  $\forall X.B \in \mathcal{U}$ .

## Appendix H. Proof of lemma 13

Let  $S_n = \Gamma \vdash \mathbf{0}:T$ . We proceed by induction over  $n$

*Basic case.*  $n = 0$ .

$$\frac{}{\Gamma \vdash \mathbf{0}:\bar{0}} ax_{\bar{0}}$$

*Inductive cases.*

1.  $\frac{\Gamma \vdash \mathbf{0}:\forall X.T}{\Gamma \vdash \mathbf{0}:T[U/X]} \forall E$  Then by the induction hypothesis  $\forall X.T \equiv \bar{0}$ , so  $T \equiv \bar{0}$  and also  $T[U/X] \equiv \bar{0}$ .
2.  $\frac{\Gamma \vdash \mathbf{0}:T}{\Gamma \vdash \mathbf{0}:\forall X.T} \forall I$  Then by the induction hypothesis  $T \equiv \bar{0}$ , so  $\forall X.T \equiv \bar{0}$ .



## Appendix I. Proof of corollary 2

Assume  $\Gamma \vdash \mathbf{t} : T$ , then by lemma 1,  $T \equiv \alpha.U$ . As  $\Gamma \vdash \mathbf{t} : \bar{0} \equiv 0.U$ , then by theorem 2,  $\alpha = 0$ .

## Appendix J. Proof of lemma 18, Item 1

Let  $SN \subseteq SN$ , we need to prove it satisfies each point of the definition of saturation.

1.  $x \mathbf{t}_1 \dots \mathbf{t}_n \in SN$ .
2. Assume  $\mathbf{v}[\mathbf{b}/x] \mathbf{t}_1 \dots \mathbf{t}_n \in SN$ , then the term

$$(\lambda x \mathbf{v}) \mathbf{b} \mathbf{t}_1 \dots \mathbf{t}_n \quad (\text{J.1})$$

must terminate because  $\mathbf{v}, \mathbf{b}, \mathbf{t}_1, \dots, \mathbf{t}_n$  terminate since these terms are  $SN$  by assumption ( $\mathbf{v}[\mathbf{b}/x]$  is a sub-term of a term in  $SN$ , hence itself is  $SN$ ; but then  $\mathbf{v}$  is  $SN$ ). After finitely many steps reducing terms in J.1 we obtain  $(\lambda x \mathbf{v}') \mathbf{b}' \mathbf{t}'_1 \dots \mathbf{t}'_n$  with  $\mathbf{v} \rightarrow^* \mathbf{v}'$ , etc. Then the contraction of  $(\lambda x \mathbf{v}') \mathbf{b}' \mathbf{t}'_1 \dots \mathbf{t}'_n$  gives

$$\mathbf{v}'[\mathbf{b}'/x] \mathbf{t}'_1 \dots \mathbf{t}'_n \quad (\text{J.2})$$

This is a reduct of  $\mathbf{v}[\mathbf{b}/x] \mathbf{t}_1 \dots \mathbf{t}_n$  and since this term is  $SN$ , also J.2 and the term  $(\lambda x \mathbf{v}) \mathbf{b} \mathbf{t}_1 \dots \mathbf{t}_n$  are  $SN$ .

3.  $\mathbf{t}, \mathbf{u} \in SN$ , then  $\mathbf{t} + \mathbf{u} \in SN$ .
4.  $\mathbf{t} \in SN$ , then  $\forall \alpha \in \mathcal{S}, \alpha.\mathbf{t} \in SN$  and vice-versa.
5.  $\forall i \in I, \mathbf{u}_i \vec{\mathbf{w}} \in SN$ , then  $(\sum_{i \in I} \mathbf{u}_i) \vec{\mathbf{w}} \rightarrow^* \sum_{i \in I} (\mathbf{u}_i \vec{\mathbf{w}})$  which is the sum of  $SN$  terms, so by 3,  $\sum_{i \in I} (\mathbf{u}_i \vec{\mathbf{w}})$  is  $SN$ . Assume  $(\sum_{i \in I} \mathbf{u}_i) \vec{\mathbf{w}}$  is not  $SN$ . Since  $\forall i, \mathbf{u}_i \vec{\mathbf{w}}$  are  $SN$ , then  $\forall i, \mathbf{u}_i$  are  $SN$ , so by 3,  $\sum_{i \in I} \mathbf{u}_i$  is  $SN$ . Let  $\mathbf{v} = (\sum_{i \in I} \mathbf{u}_i) \downarrow$ . We proceed by structural induction on  $\mathbf{v}$  to show the contradiction:
  - (a)  $\mathbf{v} = x$ , then by 1  $x \vec{\mathbf{w}}$  is  $SN$ , which is a contradiction.
  - (b)  $\mathbf{v} = \lambda x \mathbf{v}'$ , then  $\mathbf{v}'[\mathbf{w}_1/x]$  cannot be  $SN$  (in other case, by 2,  $\mathbf{v} \vec{\mathbf{w}}$  would be  $SN$ ). Notice that if  $\sum_{i \in I} \mathbf{u}_i \rightarrow^* \mathbf{v}$  with  $\mathbf{v}$  being a base term, then there are some  $i$  such that  $\mathbf{u}_i \rightarrow^* \alpha_i.\mathbf{v}$  with  $\sum_i \alpha_i = 1$ , others (may be zero)  $i$  such that  $\mathbf{u}_i \rightarrow^* \mathbf{0}$  and others which form two sets of the same size (possibly 0), one going to  $\beta.\mathbf{t}$  and the other going to  $-\beta.\mathbf{t}$ . However, by the first group, those  $\mathbf{u}_i$  going to  $\alpha_i.\mathbf{v}$ , we can see that  $\alpha_i.\mathbf{v} \vec{\mathbf{w}}$  is  $SN$ , so by 7 and 4,  $\mathbf{v} \vec{\mathbf{w}}$  is  $SN$ , which is a contradiction.
  - (c)  $\mathbf{v} = \mathbf{v}_1 \mathbf{v}_2$ . Then we can take  $\mathbf{v}_2$  as part of  $\vec{\mathbf{w}}$  and use the induction hypothesis.
  - (d)  $\mathbf{v} = \mathbf{0}$ , then it is a contradiction by item 9.
  - (e)  $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ . Since this term is in normal form, the only possibility for the reduction is  $(\mathbf{v}_1 \vec{\mathbf{w}}) + (\mathbf{v}_2 \vec{\mathbf{w}})$ , which is the sum of two terms that, by the induction hypothesis, must be  $SN$ , so the whole term is  $SN$ .
  - (f)  $\mathbf{v} = \alpha.\mathbf{v}'$ . Since this term is in normal form, the only possibility for the reduction is  $\alpha.(\mathbf{v}' \vec{\mathbf{w}})$ , which is  $SN$  by the induction hypothesis and 4.
6.  $\forall i \in I, \mathbf{u} \mathbf{v}_i \mathbf{w}_1 \dots \mathbf{w}_n \in SN$ , then  $\mathbf{u} (\sum_{i \in I} \mathbf{v}_i) \mathbf{w}_1 \dots \mathbf{w}_n \rightarrow^* \sum_{i \in I} (\mathbf{u} \mathbf{v}_i \mathbf{w}_1 \dots \mathbf{w}_n)$  which is the sum of  $SN$  terms. This case is analogous to 5.
7.  $\alpha.(\mathbf{t}_1 \dots \mathbf{t}_n) \in SN$  then  $\forall k, \mathbf{t}_1 \dots \alpha.\mathbf{t}_k \dots \mathbf{t}_n$  must terminate because  $\mathbf{t}_1, \dots, \mathbf{t}_n$  terminate since these terms are  $SN$  by assumption, so after infinitely many reduction steps reducing  $\mathbf{t}_1 \dots \alpha.\mathbf{t}_k \dots \mathbf{t}_n$  we obtain  $\alpha.\mathbf{u}$ , with  $\mathbf{t}_1 \dots \mathbf{t}_n \rightarrow^* \mathbf{u}$ . So  $\alpha.\mathbf{u}$  is a reduct of  $\alpha.(\mathbf{t}_1 \dots \mathbf{t}_n)$  and since this term is  $SN$ ,  $\forall k, \mathbf{t}_1 \dots \alpha.\mathbf{t}_k \dots \mathbf{t}_n$  are  $SN$ .
8.  $\mathbf{0} \in SN$ .
9.  $\mathbf{0} \vec{\mathbf{t}} \rightarrow^* \mathbf{0} \vec{\mathbf{t}}'$  and since  $\vec{\mathbf{t}}$  is  $SN$ , assume  $\vec{\mathbf{t}}'$  is in normal form, so  $\mathbf{0} \vec{\mathbf{t}}' \rightarrow \mathbf{0}$ , then  $\mathbf{0} \vec{\mathbf{t}} \in SN$ .
10.  $\mathbf{t} \mathbf{0} \vec{\mathbf{u}} \rightarrow^* \mathbf{t}' \mathbf{0} \vec{\mathbf{u}}'$ , as  $\mathbf{t}, \vec{\mathbf{u}}$  are in  $SN$ , assume  $\mathbf{t}', \vec{\mathbf{u}}'$  are in normal form, then  $\mathbf{t}' \mathbf{0} \vec{\mathbf{u}}' \rightarrow \mathbf{0} \vec{\mathbf{u}}' \rightarrow \mathbf{0}$ , so it is in  $SN$ .

## Appendix K. Proof of lemma 18, Item 2

Let  $U, T \in SAT$ , then  $x \in U$  by definition of saturated sets. Then  $F \in U \rightarrow T \Rightarrow F x \in T$ , as  $T \in SAT$ , then  $T \subseteq SN$ , so  $F x \in SN$  and so  $F$  is strong normalising.

$\therefore U \rightarrow T \subseteq SN$ .

Now we need to show  $U \rightarrow T$  is saturated by showing each point at the definition of saturated sets.

1. Let  $\vec{t} \in SN$ , we need to show that  $x \vec{t} \in U \rightarrow T$ , i.e.  $\forall \mathbf{b} \in U, x \vec{t} \mathbf{s} \in T$ , which is true since  $U \in SN$ , so  $\mathbf{b} \in SN$  and  $T$  is saturated, then  $x \vec{t} \mathbf{s} \in T$ .
2. Let  $\mathbf{v}[\mathbf{b}/x] \vec{t} \in U \rightarrow T$ , then  $\forall \mathbf{s} \in U, \mathbf{v}[\mathbf{b}/x] \vec{t} \mathbf{s} \in T$  and since  $T$  is saturated,  $(\lambda x \mathbf{v}) \mathbf{b} \vec{t} \mathbf{s} \in T$ , so  $(\lambda x \mathbf{v}) \mathbf{b} \vec{t} \in U \rightarrow T$ .
3. Let  $\mathbf{t}, \mathbf{u} \in U \rightarrow T \Rightarrow \forall \mathbf{s} \in U, \mathbf{t} \mathbf{s} \in T$  and  $\mathbf{u} \mathbf{s} \in T$ , then by item (e) on the definition of saturation,  $(\mathbf{t} + \mathbf{u}) \mathbf{s} \in T$ , so  $\mathbf{t} + \mathbf{u} \in U \rightarrow T$ .
4. Let  $\mathbf{t} \in U \rightarrow T$  then  $\forall \mathbf{s} \in U, \mathbf{t} \mathbf{s} \in T$ , then by the saturation of  $T$ ,  $\forall \alpha \in \mathcal{S}, \alpha.(\mathbf{t} \mathbf{s}) \in T$ , then by item (g) on the definition of saturation,  $\alpha.\mathbf{t} \mathbf{s} \in T$ , so  $\alpha.\mathbf{t} \in U \rightarrow T$ .  
Let  $\alpha.\mathbf{t} \in U \rightarrow T$ , then  $\forall \mathbf{s} \in U, \alpha.\mathbf{t} \mathbf{s} \in T$ , so by item (g) on the definition of saturation,  $\alpha.(\mathbf{t} \mathbf{s}) \in T$ , so by saturation of  $T$ ,  $\mathbf{t} \mathbf{s} \in T$ , so  $\mathbf{t} \in U \rightarrow T$ .
5. Let  $\forall i \in I, \mathbf{u}_i \vec{\mathbf{w}} \in U \rightarrow T$ , then  $\forall \mathbf{s} \in U, \mathbf{u}_i \vec{\mathbf{w}} \mathbf{s} \in T$ , then by the saturation of  $T$ ,  $(\sum_{i \in I} \mathbf{u}_i) \vec{\mathbf{w}} \mathbf{s} \in T$ , so  $(\sum_{i \in I} \mathbf{u}_i) \vec{\mathbf{w}} \in U \rightarrow T$ .
6. Let  $\forall i \in I, \mathbf{u} \mathbf{v}_i \mathbf{w}_1 \dots \mathbf{w}_n \in U \rightarrow T, \forall \mathbf{s} \in U, \mathbf{u} \mathbf{v}_i \mathbf{w}_1 \dots \mathbf{w}_n \mathbf{s} \in T$ , then by saturation of  $T$ ,  $\mathbf{u} (\sum_{i \in I} \mathbf{v}_i) \mathbf{w}_1 \dots \mathbf{w}_n \mathbf{s} \in T$ , so  $\mathbf{u} (\sum_{i \in I} \mathbf{v}_i) \mathbf{w}_1 \dots \mathbf{w}_n \in U \rightarrow T$ .
7. Let  $\alpha.(\mathbf{t}_1 \dots \mathbf{t}_n) \in U \rightarrow T$ , then  $\forall \mathbf{s} \in U, (\alpha.(\mathbf{t}_1 \dots \mathbf{t}_n)) \mathbf{s} \in T$ , then by the saturation of  $T$ ,  $\alpha.((\mathbf{t}_1 \dots \mathbf{t}_n) \mathbf{s}) \in T$ , and so, by the saturation of  $T$  again,  $\forall k, \mathbf{t}_1 \dots \alpha.\mathbf{t}_k \dots \mathbf{t}_n \mathbf{s} \in T$ , then  $\forall k, \mathbf{t}_1 \dots \alpha.\mathbf{t}_k \dots \mathbf{t}_n \in U \rightarrow T$ .  
Let  $\mathbf{t}_1 \dots \alpha.\mathbf{t}_k \dots \mathbf{t}_n \in U \rightarrow T$ , then  $\forall \mathbf{s} \in U, \mathbf{t}_1 \dots \alpha.\mathbf{t}_k \dots \mathbf{t}_n \mathbf{s} \in T$ , so  $\alpha.(\mathbf{t}_1 \dots \mathbf{t}_n \mathbf{s}) \in T$  and then  $\alpha.(\mathbf{t}_1 \dots \mathbf{t}_n) \mathbf{s} \in T$ , then  $\alpha.(\mathbf{t}_1 \dots \mathbf{t}_n) \in U \rightarrow T$ .
8. By saturation of  $T$ ,  $\forall \mathbf{s} \in U, \mathbf{0} \mathbf{s} \in T$ , then  $\mathbf{0} \in U \rightarrow T$ .
9. Let  $\vec{t} \in SN$ , then  $\forall \mathbf{s} \in U, \vec{t} \mathbf{s} \in SN$ , then by saturation of  $T$ ,  $\mathbf{0} \vec{t} \mathbf{s} \in T$ , so  $\mathbf{0} \vec{t} \in U \rightarrow T$ .
10. Let  $\mathbf{t}, \vec{\mathbf{u}} \in SN$ , then  $\forall \mathbf{s} \in U, \vec{\mathbf{u}} \mathbf{s} \in SN$ , so by the saturation of  $T$ ,  $\mathbf{t} \mathbf{0} \vec{\mathbf{u}} \mathbf{s} \in T$ , then  $\mathbf{t} \mathbf{0} \vec{\mathbf{u}} \in U \rightarrow T$ .

## Appendix L. Proof of lemma 18, Item 3

Let  $\{A_i\}_{i \in I}$  be a collection of members of  $SAT$ , then  $\forall i \in I, A_i \subseteq SN$ , so  $\bigcap_{i \in I} A_i \subseteq SN$ .

We have to show that  $\bigcap_{i \in I} A_i$  is saturated.

1.  $\forall i \in I, \forall n \geq 0, x \mathbf{t}_1 \dots \mathbf{t}_n \in A_i$  where  $\mathbf{t}_i \in SN$  and  $x$  is any term variable, then  $x \mathbf{t}_1 \dots \mathbf{t}_n \in \bigcap_{i \in I} A_i$
2.  $\forall n \geq 0$ , let  $\mathbf{v}[\mathbf{b}/x] \mathbf{t}_1 \dots \mathbf{t}_n \in \bigcap_{i \in I} A_i$ , then  $\forall i \in I, \mathbf{v}[\mathbf{b}/x] \mathbf{t}_1 \dots \mathbf{t}_n \in A_i$ , so  $\forall i \in I, (\lambda x \mathbf{v}) \mathbf{b} \mathbf{t}_1 \dots \mathbf{t}_n \in A_i$  and then  $(\lambda x \mathbf{v}) \mathbf{b} \mathbf{t}_1 \dots \mathbf{t}_n \in \bigcap_{i \in I} A_i$ .
3. Let  $\mathbf{t}, \mathbf{u} \in \bigcap_{i \in I} A_i$ , then  $\forall i \in I, \mathbf{t}, \mathbf{u} \in A_i$  so  $\forall i \in I, \mathbf{t} + \mathbf{u} \in A_i$  and then  $\mathbf{t} + \mathbf{u} \in \bigcap_{i \in I} A_i$ .
4.  $\mathbf{t} \in \bigcap_{i \in I} A_i \Leftrightarrow \forall i \in I, \mathbf{t} \in A_i \Leftrightarrow \forall \alpha \in \mathcal{S}, \forall i \in I, \alpha.\mathbf{t} \in A_i \Leftrightarrow \forall \alpha \in \mathcal{S}, \alpha.\mathbf{t} \in \bigcap_{i \in I} A_i$ .
5.  $\forall j \in J, \mathbf{u}_j \vec{\mathbf{w}} \in \bigcap_{i \in I} A_i \Rightarrow \forall i \in I, \forall j \in J, \mathbf{u}_j \vec{\mathbf{w}} \in A_i \Rightarrow \forall i \in I, (\sum_{j \in J} \mathbf{u}_j) \vec{\mathbf{w}} \in A_i \Rightarrow (\sum_{j \in J} \mathbf{u}_j) \vec{\mathbf{w}} \in \bigcap_{i \in I} A_i$ .
6.  $\forall j \in J, \mathbf{u} \mathbf{v}_j \vec{\mathbf{w}} \in \bigcap_{i \in I} A_i \Rightarrow \forall i \in I, \forall j \in J, \mathbf{u} \mathbf{v}_j \vec{\mathbf{w}} \in A_i \Rightarrow \forall i \in I, \mathbf{u} (\sum_{j \in J} \mathbf{v}_j) \vec{\mathbf{w}} \in A_i \Rightarrow \mathbf{u} (\sum_{j \in J} \mathbf{v}_j) \vec{\mathbf{w}} \in \bigcap_{i \in I} A_i$ .
7.  $\alpha.(\mathbf{t}_1 \dots \mathbf{t}_n) \in \bigcap_{i \in I} A_i \Leftrightarrow \forall i \in I, \alpha.(\mathbf{t}_1 \dots \mathbf{t}_n) \in A_i \Leftrightarrow \forall i \in I, \mathbf{t}_1 \dots \alpha.\mathbf{t}_k \dots \mathbf{t}_n \in A_i$  with  $(1 \leq k \leq n) \Leftrightarrow \mathbf{t}_1 \dots \alpha.\mathbf{t}_k \dots \mathbf{t}_n \in \bigcap_{i \in I} A_i$ .
8.  $\forall i \in I, \mathbf{0} \in A_i$ , then  $\mathbf{0} \in \bigcap_{i \in I} A_i$ .
9.  $\forall i \in I, \forall \vec{t} \in SN, (\mathbf{0} \vec{t}) \in A_i$ , so  $(\mathbf{0} \vec{t}) \in \bigcap_{i \in I} A_i$ .
10.  $\forall i \in I, \forall \mathbf{t}, \vec{\mathbf{u}} \in SN, (\mathbf{t} \mathbf{0}) \vec{\mathbf{u}} \in A_i$ , then  $(\mathbf{t} \mathbf{0}) \vec{\mathbf{u}} \in \bigcap_{i \in I} A_i$ .

## Appendix M. Proof of lemma 18, Item 4

By structural induction on  $T$ .

$T := X$ . Then  $\llbracket T \rrbracket_\xi = \xi(X) \in SAT$ .

$T := A \rightarrow B$ . Then  $\llbracket T \rrbracket_\xi = \llbracket A \rrbracket_\xi \rightarrow \llbracket B \rrbracket_\xi$ . by the induction hypothesis  $\llbracket A \rrbracket_\xi, \llbracket B \rrbracket_\xi \in SAT$ , then by lemma 18(2),  $\llbracket A \rrbracket_\xi \rightarrow \llbracket B \rrbracket_\xi \in SAT$ .

$T := \forall X.T'$ . Then  $\llbracket T \rrbracket_\xi = \bigcap_{Y \in SAT} \llbracket T' \rrbracket_{\xi(X:=Y)}$ . by the induction hypothesis  $\forall Y \in SAT, \llbracket T' \rrbracket_{\xi(X:=Y)} \in SAT$ , then by lemma 18(3),  $\bigcap_{Y \in SAT} \llbracket T' \rrbracket_{\xi(X:=Y)} \in SAT$ .

## Appendix N. Proofs of intermediate results in the proof of theorem 6

**R1:** Structural induction on  $\mathbf{t}_1$ .

*Basic cases.*

1.  $\mathbf{t}_1 = x$ . Done.
2.  $\mathbf{t}_1 = \lambda x \mathbf{u}$ , then  $\mathbf{t}_1 \mathbf{t}_2 \rightarrow \mathbf{u}[\mathbf{t}_2/x]$ , which is a contradiction.
3.  $\mathbf{t}_1 = \mathbf{0}$ , then  $\mathbf{t}_1 \mathbf{t}_2 \rightarrow \mathbf{0}$ , which is a contradiction.

*Inductive cases.*

1.  $\mathbf{t}_1 = \alpha.\mathbf{u}$ , then  $\mathbf{t}_1 \mathbf{t}_2 \rightarrow \alpha.(\mathbf{u} \mathbf{t}_2)$ , which is a contradiction.
2.  $\mathbf{t}_1 = \mathbf{u} + \mathbf{v}$ , then  $\mathbf{t}_1 \mathbf{t}_2 \rightarrow (\mathbf{u} \mathbf{t}_2) + (\mathbf{v} \mathbf{t}_2)$ , which is a contradiction.
3.  $\mathbf{t}_1 = (\mathbf{u} \mathbf{v})$ , then by the induction hypothesis there are two options:
  - (a)  $\mathbf{u} = x$ , so  $\mathbf{u} \mathbf{v} = x \vec{\mathbf{r}}$  where  $\vec{\mathbf{r}} = \mathbf{v}$ , or
  - (b)  $\mathbf{u} = x \vec{\mathbf{r}}$ , so  $\mathbf{u} \mathbf{v} = x \vec{\mathbf{s}}$ , where  $\vec{\mathbf{s}} = \vec{\mathbf{r}} \mathbf{v}$ .

**R2:** Let  $S_n = \Gamma \vdash x : T$ . Induction over  $n$ .

*Basic case.  $n = 0$ .*

$$\frac{}{\Gamma, x : T \vdash x : T} ax$$

Then  $T \in \mathcal{C}$ , as contexts have only classic types in the type system  $\mathcal{P}$ .

*Inductive cases.*

1.  $\frac{\Gamma \vdash x : \forall X.T}{\Gamma \vdash x : T[C/X]} \forall E[X := C]$  Then by the induction hypothesis  $\forall X.T \in \mathcal{C}$ , so  $T \in \mathcal{C}$  and then  $T[C/X] \in \mathcal{C}$ .
2.  $\frac{\Gamma \vdash x : T}{\Gamma \vdash x : \forall X.T} \forall I$  Then by the induction hypothesis  $T \in \mathcal{C}$ , so  $\forall X.T \in \mathcal{C}$ .

**R3:** Let  $S_n = \Gamma \vdash x \vec{\mathbf{r}} : T$ . Induction over  $n$ .

*Basic case.  $n = 1$ .* Notice that  $\vec{\mathbf{r}}$  cannot be  $\mathbf{0}$ , as  $(x \mathbf{0}) \rightarrow \mathbf{0}$ . So, the only possibility is

$$\frac{\frac{}{\Gamma, x : U \rightarrow T, y : U \vdash x : U \rightarrow T} ax \quad \frac{}{\Gamma, x : U \rightarrow T, y : U \vdash y : U} ax}{\Gamma, x : U \rightarrow T, y : U \vdash x y : T} \rightarrow E$$

Then, by **R2**,  $U \rightarrow T \in \mathcal{C}$ , so  $T \in \mathcal{C}$ .

*Inductive cases.* Notice that  $x \vec{\mathbf{r}} = (((x \mathbf{r}_1) \mathbf{r}_2) \dots \mathbf{r}_n)$

1. 
$$\frac{\Gamma \vdash (x \vec{r}) : \alpha.(U \rightarrow T) \quad \Gamma \vdash \mathbf{t} : \beta.U}{\Gamma \vdash (x \vec{r}) \mathbf{t} : (\alpha \times \beta).T} \rightarrow E$$
 Then by the induction hypothesis,  $U \rightarrow T \in \mathcal{C}$ , so  $T \in \mathcal{C}$  and  $T \equiv 1.T$ .
2. 
$$\frac{\Gamma \vdash x \vec{r} : \forall X.T}{\Gamma \vdash x \vec{r} : T[C/X]} \forall E$$
 Then by the induction hypothesis  $\exists D \in \mathcal{C}, \alpha \in \mathcal{S}$  such that  $\forall X.T \equiv \alpha.D$ , so  $\exists E \in \mathcal{C}$  s.t.  $T \equiv \alpha.E$ , then  $T[C/X] \equiv (\alpha.E)[C/X] \equiv \alpha.E[C/X]$ . Notice that  $E[C/X] \in \mathcal{C}$ .
3. 
$$\frac{\Gamma \vdash x \vec{r} : T}{\Gamma \vdash x \vec{r} : \forall X.T} \forall I$$
 Then by the induction hypothesis  $\exists C \in \mathcal{C}, \alpha \in \mathcal{S}$  s.t.  $T \equiv \alpha.C$ , so  $\forall X.T \equiv \forall X.\alpha.C \equiv \alpha.\forall X.C$ .